Spontaneous Focusing on Quantitative Relations as a Predictor of the Development of Rational Number Conceptual Knowledge

Jake McMullen, Minna M. Hannula-Sormunen, Eero Laakkonen & Erno Lehtinen

University of Turku

Jake McMullen, Department of Teacher Education and Centre for Learning Research, University of Turku, Finland; Minna M. Hannula-Sormunen, Department of Teacher Education and Turku Institute of Advanced Studies, University of Turku, Finland; Eero Laakkonen, Department of Teacher Education, University of Turku, Finland; Erno Lehtinen, Department of Teacher Education and Centre for Learning Research, University of Turku, Finland.

This research was supported by Academy of Finland grant 131490 to the third author. We thank Satu Katajisto and Milla Laaksonen for help with data collection and the teachers and students for their collaboration.

Correspondence concerning this article should be addressed to Jake McMullen, Centre for Learning Research, Turun Yliopisto, 20014, Finland. Email: jake.mcmullen@utu.fi

Date Accepted: September 29, 2015

Abstract

Many people have serious difficulties in understanding rational numbers, limiting their ability to interpret and make use of them in modern daily life. This also leads to later difficulties in learning more advanced mathematical content. In this study, novel tasks are used to measure 263 late primary school students' spontaneous focusing on quantitative relations, in situations that are not explicitly mathematical. Even after controlling for a number of known predictors of rational number knowledge, spontaneous focusing on quantitative relations is found to have a strong impact on students' learning of rational number conceptual knowledge. This finding opens new possibilities for developing pedagogical solutions to one of the most difficult challenges of mathematics education. The findings suggest that students' own focusing tendency and self-initiated practice may have on important role in the long-term development of complex cognitive skills.

Spontaneous Focusing on Quantitative Relations as a Predictor of the Development of Rational Number Conceptual Knowledge

Educators and researchers have put a great deal of attention in recent years on the topic of rational numbers in mathematics education, both in terms of students' difficulties with learning the topic and the key role rational number knowledge plays in the development of advanced mathematical competences (Siegler, Thompson, & Schneider, 2011; Van Dooren, Lehtinen, & Verschaffel, 2015; Vosniadou & Verschaffel, 2004). Rational numbers (fractions, fractional decimals, and percentages) also play an important role in decision making and reasoning in a variety of everyday situations (Reyna & Brainerd, 2007). As such, failure to adequately develop rational number knowledge not only affects later learning in mathematics (Siegler et al., 2012), but can have long-lasting effects on work-life. The aim of the present study is to investigate a novel predictor of individual differences in rational number learning, namely how students spontaneously focus on quantitative relations.

Theoretical Framework

The Development of Rational Number Conceptual Knowledge

Relatively early on, young children are able to display adequate proportional reasoning and can understand many types of quantitative relations in practical situations (Boyer, Levine, & Huttenlocher, 2008; Singer-Freeman & Goswami, 2001; Sophian, 2000; Spinillo & Bryant, 1999). Despite this, when fractions are formally taught in school most students have severe difficulties in the extension of the number concept from natural numbers to fractions and fractional decimals (McMullen, Laakkonen, Hannula-Sormunen, & Lehtinen, 2015b; Ni & Zhou, 2005; Stafylidou & Vosniadou, 2004; Vamvakoussi, Chrisou, Mertens, & Van Dooren, 2011; Vamvakoussi & Vosniadou, 2004, 2010). These difficulties are especially apparent with the knowledge of the magnitude of fractions and decimals – represented in symbolic form – and the dense nature of the set of rational numbers. These difficulties in understanding rational number concepts are surprisingly resistant to teaching (Fuchs et al., 2013; Merenluoto & Lehtinen, 2004; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Vamvakoussi & Vosniadou, 2010; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013), and have more generic and long-lasting consequences than shortages in procedural knowledge (Bempeni & Vamvakoussi, 2015; Hallett, Nunes, & Bryant, 2010; Hecht & Vagi, 2012). In many situations, natural number knowledge is not sufficient for reasoning about rational numbers, and can be even detrimental. Expanding the number concept to include features of rational numbers that are incongruent with previously held concepts of natural numbers may be hampered by the so-called natural (or whole) number bias (Ni & Zhou, 2005), in which features of natural numbers are incorrectly used in reasoning about rational numbers. It has been argued that in order to overcome such a natural number bias, substantial conceptual change may be needed (McMullen et al., 2015b; Vamvakoussi & Vosniadou, 2010; Vosniadou, 2014).

One extant theory is that concepts of the magnitude of numbers provide a developmental bridge between whole and rational numbers (Siegler et al., 2012), with some evidence suggesting that early whole number knowledge predicts later rational number knowledge (Bailey, Siegler, & Geary, 2014; Jordan et al., 2013; Vukovic et al., 2014). However, when examined concurrently among late primary school children, the relation between whole number knowledge and rational number knowledge is mitigated by overall mathematical achievement (Van Hoof, 2015). Together, these findings suggest that prior knowledge of natural numbers is relevant, but insufficient, for explaining developmental differences in rational number knowledge.

If formal knowledge of natural numbers cannot fully explain later learning of rational numbers, it is important to investigate other features of students’ mathematically relevant experiences that may contribute to the extension of number concept from natural to rational numbers. Recent research has examined differences in how children spontaneously focus on mathematically relevant aspects of their environment (e.g. numerosity or quantitative relations) and make use of them in action (Hannula &

Lehtinen 2005; Hannula, Lepola & Lehtinen, 2010; Hannula, Räsänen & Lehtinen, 2007; Hannula-Sormunen, 2014; Hannula-Sormunen, Lehtinen & Räsänen, 2015; McMullen, 2014; McMullen, Hannula-Sormunen, & Lehtinen, 2011, 2013, 2014, 2015a) or in verbal descriptions (Batchelor, 2014). It is apparent that not all children and students equally focus on the numerosities or quantitative relations that exist in everyday situations, when these features are not explicitly highlighted, even when they have the ability to do so. Recent studies have explored how these individual differences in spontaneous quantitative focusing tendencies are related to later learning of mathematical knowledge (e.g., Hannula-Sormunen, Lehtinen, & Räsänen, 2015), including fractions (McMullen et al., 2014, 2015a).

Spontaneous Focusing On quantitative Relations (SFOR)

The origins of SFOR lie in earlier studies on the tendency of Spontaneous Focusing On Numerosity (SFON), which has been identified as a predictor of mathematical skills in normally developing (Batchelor, 2014; Edens & Potter 2013; Hannula & Lehtinen, 2005; Hannula et al., 2010; Hannula-Sormunen et al., 2015; Potter, 2009) and dyscalculic children (Kucian et al., 2012). SFON refers to spontaneously focusing attention on the exact number of a set of items or incidents when exact numerosity is utilized in action (Hannula & Lehtinen, 2005; Hannula et al., 2010). This distinguishable attentional process is needed for triggering exact number recognition and the subsequent use of the recognized exact number in action. Measures of SFON tendency are an indicator of the amount spontaneous practice with exact enumeration in natural surroundings (Hannula & Lehtinen, 2005). Individual differences in SFON tendency have not been explained by motivational factors (Edens & Potter, 2013), nor by general attentional skills (Hannula et al., 2010), non-verbal IQ, or comprehension of instructions (Hannula & Lehtinen, 2005; Hannula et al., 2010). SFON tendency in preschool predicts mathematical skills up to six years later, even when subitizing-based enumeration, verbal number skills, and nonverbal IQ are controlled for (Hannula-Sormunen et al., 2015).

There is preliminary evidence to suggest that a strong SFON tendency during pre-school predicts students' learning of fractions when they are first introduced in school (McMullen et al., 2015a). However, in many everyday activities, exact numerosity is not the only mathematically relevant aspect that can be focused on. Imagine a seven-year-old child traveling with her mother to visit their grandparents in the countryside. During the boring car drive the child starts to spontaneously to think about the trip in terms of quantitative relations, asking "Are we halfway there yet?". Similar reasoning can occur during lunch ("I have half of my broccoli left") or reading a book ("This book is twice as long as the last book I read!"). All of these descriptions require the recognition and use of mathematical, or quantitative, relations, and often this is done without explicit guidance to do so.

Recently, SFOR tendency has been introduced as a potential source of individual differences in mathematical development (McMullen et al., 2011, 2013), as measured by action-based imitation tasks in 5- to 7-year-old children. SFOR tendency is defined as the spontaneous, (i.e. unguided) focusing of attention on quantitative relations and the use of these relations in non-explicitly mathematical situations. SFOR tendency involves focusing on the quantifiable relation(s) between of two or more (sub-)sets of items or quantities, rather than only focusing on the numerosity of separate items or quantifiable amounts (McMullen et al., 2014). For example, instead of focusing on the number of different apples in the bowl, such as two green apples and four red apples, focusing on the quantitative relations of the different kinds of apples, such as there are twice as many red apples as green apples or that one third of the apples are green.

In studies of young children in the United States and Finland, SFOR was considered a distinguishable cognitive and attentional tendency (McMullen et al., 2011, 2013, 2014). Thus, individual differences in SFOR are not entirely explained by individual differences in the skills needed to recognize quantitative relations or other cognitive abilities required to solve the tasks (McMullen et al., 2014). If different practical situations can trigger reasoning about quantitative relations depending on the strength of SFOR tendency of the students, it can be expected that students with a strong SFOR tendency acquire more, and more varied, self-initiated practice with quantitative and numerical relations than students with a lower SFOR tendency. Self-initiated practice of this sort may provide valuable experiences that support these students' mathematical development, including the development of rational number conceptual knowledge (McMullen, Hannula-Sormunen & Lehtinen, 2015a).

The Present Study

The present study aims to extend these previous preliminary findings that suggest that SFOR may play a role in early primary school by investigating if SFOR tendency predicts late primary school students' learning of rational number conceptual knowledge during a one-and-a-half year period when fractions and fractional decimals are first taught at school. According to the Finnish National Curriculum, fractions and decimals are taught in the 3rd to 6th grades. In order to gain a comprehensive outlook on the role of SFOR tendency in learning of rational numbers, students from grades three through five will be followed over two school years, covering the whole instructional period in primary school. Previous studies of spontaneous quantitative focusing tendencies have highlighted the importance of carefully controlling whether individual differences in the cognitive or mathematical skills required to solve the SFON or SFOR tasks explain the test results (Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; McMullen et al., 2014). The present study is the first which manages to create an indicator of pure SFOR tendency by explicitly removing the variance explained by individual differences in the ability to recognize and use quantitative relations in the SFOR tasks. This is done by using tasks that were similar to SFOR tasks, but in which students' attention is explicitly guided to the aspect of quantitative relations, and thus variance in these guided tasks are due to differences in the ability to recognize quantitative relations along with the other cognitive skills needed in the tasks. By removing this variance it is possible to determine the specific role SFOR tendency plays in the development of rational number conceptual knowledge.

First, individual differences in SFOR tendency are investigated. Previous studies have found substantial individual differences in SFOR tendency among young children from ages 5 to 10, in both the US and Finland (McMullen et al., 2013, 2014). Among younger children, individual differences in SFOR were not explained by the ability to recognize and use quantitative relations in tasks in which participants were explicitly guided to use the relations to solve the task (McMullen et al., 2014). We therefore expect there to be individual differences in SFOR tendency in late primary school children, which are not entirely explained by differences in students' ability to recognize and describe quantitative relations (Hypothesis 1).

Second, we examine the unique contribution of a measure of "pure" SFOR tendency in explaining the development of rational number conceptual knowledge, beyond grade level, whole number size knowledge, non-verbal intelligence, and whole number arithmetic. For the first time, such a measure of "pure" SFOR tendency is used, which removes the variance in SFOR task responses that is explained by other cognitive abilities needed in the task, including the ability to recognize quantitative relations when explicitly guided to do so. Previous studies have found a link between early whole number magnitude knowledge and arithmetic skills and later rational number knowledge (e.g. Bailey et al., 2014). Early skills dealing with quantitative relations, measured by explicitly mathematical tasks, have been shown to predict later rational number knowledge (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009; Möhring, Newcombe, Levine, & Frick, 2015). However, earlier studies on the long term contributions of spontaneous quantitative focusing tendencies to mathematical development in younger children suggest that SFOR may have a sizable effect on rational number conceptual development. Thus, it is expected that SFOR tendency provides a unique contribution to the development of conceptual knowledge of rational numbers during the follow-up period. (Hypothesis 2).

Methods

Participants and Procedures

Students from six classes at two schools in southwest Finland (N = 263; 141 females) participated in the study. At the start of the present study participants were in Grade Three (n = 89; M_{age} = 9 years; 7 months; *SD* = 4 months), Grade Four (n = 81; M_{age} = 10 years; 8 months, *SD* = 4 months), and Grade Five (n = 93; M_{age} = 11 years; 6 months, *SD* = 4 months). The sample was representative of the Finnish urban population, including students from lower-middle class to middle class backgrounds and from diverse ethnic backgrounds (80% of participants had a Finnish background and 20% non-Finnish background, primarily from the Middle East, North Africa, Russia, and Southeast Asia). All participants had parental permission to participate in the study and ethical board approval was granted for this study.

Altogether, participants completed tasks at four time points. There were 12 participants (5 Grade Three, 6 Grade Four, and 1 Grade 5) who did not participate at the second time point (3 of whom missed all subsequent time points), 30 participants (12 Grade Three, 5 Grade Four, and 13 Grade Five) did not participate at the third time point (8 of whom missed all subsequent time points), and 24 at the fourth time point (9 Grade Three, 7 Grade Four, and 8 Grade Five). Those who did not participate at a time point were absent from school at the time of the assessments, for reasons not related to data collection (e.g. illness, moved away) and did not differ from the rest of the sample with regards to initial knowledge of rational number concepts (Time 2: t (261) = 0.81, p = 0.42; Time 3: t (261) = 0.52, p = 0.60; Time 4: t (261) = 1.70, p = 0.10).

Students completed a test of rational number conceptual knowledge a total of 4 times over the course of two school years, at the beginning and end of the spring semester, and again at the beginning and end of the next years' spring semester. Time 1 was in February 2012, Time 2 was in May 2012, Time 3 was in January 2013, and Time 4 was in May 2013. Data collections were either prior to (Times

1 and 3) or after (Times 2 and 4) participants' regular course on fractions and decimals in the spring semesters. Students in all three grades had regular fraction and decimal courses covering basic topics such as learning about the representations of rational numbers, including the size of fractions and decimals, basic arithmetic computation with fractions and decimals, and number line estimation. There was no direct instruction on the density of rational numbers during the basic courses. However, students were indirectly confronted with concepts of density, especially that there can always be decimals or fractions between given numbers on the number line, when learning about the base-10 system in the construction of decimal numbers and exercises involving number line estimation. Students were also taught that fractions can always be further broken down when dividing "pizza" models in smaller units or finding common denominators. While the instructional depth and level of difficulty in rational number topics naturally increases from Grades 3 to 5 in the Finnish curriculum, the timing and amount of content covered remains fairly stable.

Two trained research assistants presented all tasks in the students' home classroom, in a timed whole classroom setting.

SFOR tasks

At Time 1 participants completed measures of SFOR tendency. The participants were not made aware of the mathematical nature of the SFOR tasks. Prior to and while completing the tasks, the researchers made no mention of any mathematical aspects. Testing was completed during a class period that was not mathematics. All the SFOR tasks were picture description tasks, in which the students were asked to describe either the location of a vehicle (Road task), how objects had changed during a transformation (Teleportation task), or what was on a plate of food (Plate task). Images were presented on a screen at the front of the classroom and on paper on which the students recorded their answers. All directions were read aloud to the whole class and written on the forms.

Road Task. In this task, a picture of a curved road was shown with a city landscape at the beginning and a cottage at the end, at approximately ¼, ½, and ¾ of the length of the road there were images of a river, a forest, and mountains. Participants were told that there were many cars travelling on this road. They were then shown the road with a black bus approximately half-way from the city to the cottage, next to the forest, and were asked to describe, "in as many ways as possible", where the vehicle was on its journey. Two more trials showed a car and a truck at one-quarter and three-quarters of the distance respectively. In all, there were three items on the road task.

-Insert Figure 1 about here-

Teleportation Task. The teleportation task involved a cover story about space colonies that were sent material via teleportation devices, which sometimes changed the material. Students were shown the original material and then the material that had arrived at the space colony (See Figure 1). The materials had all changed color, shape, and number of items in a uniform way (e.g. all shades of red became shades of blue, they were all elongated, the number of objects was divided by two). On the first trial, participants were asked to describe, "in as many ways as possible", how the material had changed during teleportation. On the second trial, they were shown a different amount of the same original objects and were asked to draw what they would expect to arrive based on the previous time. They then repeated these trials with new material that was common food items, which changed color, type (e.g. milk to juice), and number of objects. In total, there were four items on the teleportation task.

-Insert Figure 2 about here-

Plate Task. This task consisted of images of plates of food that were proportionally arranged on a plate (See Figure 2), and it was stated that a balanced meal is important for healthy living. Participants were then asked to describe, "in as many ways as possible", the meal in the picture. All together four items were presented on the plate task.

SFOR analysis. Participants' responses were scored on each trial separately from zero to two points based on whether or not, and how precisely, they described quantitative relations in their responses. Since participants were not guided towards the mathematical features of the tasks, it is claimed that if they described quantitative relations in their responses, they needed to first pay attention to the quantitative relations in the tasks. In other words, using quantitative relations in a response on these tasks requires the participant to spontaneously focus on quantitative relations on that trial.

Two points were given for descriptions of exact proportional or multiplicative relations (e.g. "It's all split into thirds"; "The car is half-way"). For the drawing items on the teleportation task, two points were given for drawings that depicted the correct number of items for all three types of objects based on the multiplicative relation from the previous item (e.g. all items multiplied by 3).

One point was given for descriptions that included non-exact descriptions of mathematical or multiplicative relations ("They multiplied") or an incorrect proportional relation. For the teleportation drawing items, one point was given for drawing two out of three objects correctly or drawing a consistent, but incorrect, multiplicative change (e.g. all items were multiplied by 2). All other responses were considered non-relational and given a score of zero.

The maximum raw SFOR scores were 6 for the road task, 8 for the teleportation, and 8 for the plate task. Two independent raters coded 29% of the items; agreement was found on 98% of these items.

Raw SFOR score. In order to acquire a more generalizable measure of SFOR tendency across several different SFOR tasks, a SFOR tendency sum score was created by applying item analysis aimed at identifying an internally consistent score from all trials in the SFOR tasks (Prieto, Alonso, & Lamarca, 2003). Observing the corrected item-total correlations for all items across the three tasks, items on the road task had particularly low correlation coefficients with the items from the other tasks (r < .19). Subsequently, one-by-one, the item that would most improve the overall reliability was removed, until the reliability could not be improved any more. Finally, all eight of the teleportation and plate items were included as the final SFOR sum variable, with the Road task items (M = 1.13, SD = 1.62) all excluded from the measure. The maximum possible score was 16. The resulting reliability was relatively high with an average measure intra-class correlation = .80.

Rational Number Test

Core Rational Number Test. The base of the Rational Number Test used at all four time points, consisted of 22 multiple choice and short answer problems based on tasks used in earlier studies (Martinie, 2007; Stafylidou & Vosniadou, 2004). Students had 15 minutes to complete this portion of the test. The test consisted of three types of items that measured students' conceptual knowledge of the size of rational numbers and the density of rational numbers. Size items consisted of the comparison of fractions and/or decimals and the ordering of fractions or decimals. Size items were always incongruent with regard to natural number features and rational number features, meaning that they could not be correctly solved by direct comparison of the component natural numbers. Density items consisted of questions about the sequence of rational numbers and the largest and smallest possible fractions.

Comparison items were multiple-choice and were made up of three fraction comparison items (e.g. "Circle the larger fraction. If the numbers are equal circle both." 5/8; 4/3), three decimal comparison items (e.g. "Circle the larger decimal…": 0.36; 0.5), and four items comparing fractions and decimals (e.g. "Circle the large number…": 1/8; 0.8). Ordering items were short answer responses including: three fraction ordering items (e.g. "Put the numbers in order from smallest to largest": 6/8; 2/2; 1/3) and three decimal ordering items (e.g. "Put the numbers in order from smallest to largest": 6.79; 6.786; 6.4). Each item was scored as correct or incorrect with a maximum score of 16 for the size portion of the test.

Density items required short answer responses for two items on the density of fractions (e.g. "Are there other fractions between 3/5 and 4/5? If there are, then how many?"), two items on the density of decimals (e.g. “Are there other decimal numbers between 0.3 and 0.4? If there are, how many?”), and two items on fraction infinity concepts (e.g. “What is the smallest possible fraction?”). Each item was scored as incorrect (0 points), partially correct (1 point), or correct (2 points). Correct responses were those that displayed a mathematically correct concept of the density or the infinite nature of rational numbers, typically by stating that there are an infinite number of numbers between any two rational numbers or that it’s impossible to say how many there are (e.g. “There are an infinite number [of fractions between 3/5 and 4/5.]”). Partially correct responses included some understanding that there are a very large number of rational numbers in between any two other rational numbers, but did not include notions of infinity (e.g. “many”). Incorrect responses displayed no understanding of the density of rational numbers, stating that there are only one or no number(s) in between any two numbers (e.g. “There are no numbers [in between 0.3 and 0.4.]”). The maximum score for the density portion of the test was 12, indicating six correct answers.

The basic format of the Rational Number Test remained the same across all measurement points, though the actual items were slightly altered (e.g. 3/2 and 5/7 became 4/3 and 5/8) for each subsequent testing to lower test-retest effects. Two independent raters scored 25% of the density items from the testing at Time 1 and agreed on 96% of them. Thus, this coding scheme was followed for the rest of the testing. The maximum score for the whole test was 28, reliability across all items was high (Cronbach’s alpha = .89).

Extra density items. Due to time constraints at the first measurement point, when the SFOR tasks were also given, extra density items were included only at Time 4. These were aimed at capturing more variation in emerging density knowledge (items were based on Vamvakoussi and Vosniadou, 2010). Along with the six density items already included in the basic version of the Rational Number Test, participants also completed four multiple choice adaptations of the density questions (e.g. “How many numbers are there in between 1/3 and 2/3?”). Possible responses were, a) There are no numbers,

b) There is [e.g. 3/6], c) There are an infinite number of [fractions], d) There are an infinite number of numbers, including fractions and decimals. As well, two open-ended items where participants were asked to place a fraction (1/5) or decimal (0.56) on a number line with an adjacent fraction (2/5) or decimal (0.55) already located on the number line. After which the density question using these two numbers was asked. The same coding scheme was used to score these additional density items as in the previous density items, thus the maximum score for this portion of the density items was 6. Cronbach's alpha was .87 for these items.

Control Measures

Guided focusing on quantitative relations. At Time 1, in order to determine if, when explicitly guided to do so, participants were able to recognize quantitative relations and describe them, the first trial from the teleportation and plate tasks were repeated after the completion of all SFOR tasks, with explicit instructions to describe the quantitative relations in the situation. Responses were scored, based on a simplified version of the criteria used in scoring the corresponding SFOR tasks, as either a) 1 point for a relational response, if they were able to provide a response that was at least as accurate as the level of 1-point responses on the original SFOR tasks, or b) 0 points for non-relational responses. The maximum possible score on the task was 2, with a reliability of $\alpha = .51$.

Arithmetic fluency. At Time 1, the Math Fluency test of the Woodcock-Johnson Tests of Achievement (WJ-III®) was administered. Participants completed as many arithmetic problems (addition, subtraction, multiplication, division) as possible in 3 minutes. The possible maximum was 160 correct responses. The reliability and validity of the WJ-III has been comprehensively reported previously (Woodcock, McGrew, Mather, 2001).

Non-verbal intelligence. At Time 1, Raven's Colored Progressive Matrices (Raven, 1976) were used as a measure of non-verbal intelligence. A total of 20 items were used, including two items from Set B (B1 and B2), the whole of Set C (C1 – C12), and six items from Set D (D1 – D6). Items were scored as correct or incorrect with the maximum score of 20. Reliability remained fairly high for the items, Kuder-Richardson = .78.

Number line estimation. Due to time constraints based on the agreement with schools an measure of whole number estimation was included at Time 2. Participants completed a paper-and-pencil measure of whole number estimation, by estimating the location of 500, 90, and 720 on a 15.3 cm long number line with endpoints of 0 and 1000. Participants' scores were the average difference between their response and the correct placement multiplied by -1 to adjust for a larger distance being less correct (Bailey et al., 2014).

Statistical Analysis

Standard ANOVA, correlation, and regression analyses were run using IBM SPSS (Version 21) in order to investigate differences in SFOR tendency and how these are related to later rational number conceptual knowledge. In order to investigate how SFOR tendency predicts the development of rational number conceptual knowledge a latent growth curve model (LGCM) of rational number conceptual knowledge with SFOR tendency as a predictor was estimated. A LGCM is able to estimate both the initial performance, indicated by an intercept (I) variable, and the growth over multiple time points from this initial performance, indicated by a slope (S) variable. Slope variables can be linear or curvilinear, however, in the present study the most parsimonious model, using a linear slope, will be first considered.

Intercept and Slope variables were estimated for each participant to allow for the investigation of individual differences in development. In this way, it is possible to measure how SFOR tendency was related to both the intercepts and slopes for rational number conceptual knowledge. The LGCM was estimated using Mplus 7.0 (Muthen & Muthen, 1998-2012). The estimation method was maximum likelihood with robust standard errors (MLR), which, as a full information approach, can handle missing-at-random data. Chi-square test and fit indices were used to determine goodness of fit of the

LGCM. The comparative fit index (CFI) threshold was set at > .95, root-mean-square error of approximation (RMSEA) was set at < .08, and standardized root mean square residual (SRMR) at < .05 (Byrne, 2012; Hu & Bentler, 1999)

Results

Individual and Grade-Level Differences in SFOR Tendency

Table 1 details the descriptive statistics for the predictor and outcome variables. Table 2 provides a breakdown of descriptive statics by grade level). These results indicate that there are substantial individual differences in raw SFOR and some differences in the ability to successfully use quantitative relations in the guided focusing on quantitative relations tasks. In total, only 5.2% of participants were unable to describe the quantitative relations on at least one trial on either the spontaneous or guided versions of the tasks.

-Insert Table 1 about here-

-Insert Table 2 about here-

Residualized SFOR scores were calculated from a regression of raw SFOR scores on guided focusing on quantitative relation scores: F (1, 249) = 70.95, p < .001, R^2 = .22. These residualized SFOR scores are an indication of participants' SFOR tendency in relation to what is expected based on their guided focusing on quantitative relational responses in comparison to the rest of the sample (Hallett et al., 2010). This procedure allows for a "pure" SFOR measure that only reflects individual differences in students' SFOR tendency, and not individual differences in students' other cognitive abilities needed for describing quantitative relations in the SFOR tasks.

-Insert Table 3 about here-

As can be seen in Table 3, when looking within grade levels, substantial individual differences in residualized SFOR scores appear, suggesting that individual differences in raw SFOR are not due solely to differences in the ability to recognize and describe quantitative relations in the SFOR tasks.

Absolute values of skewness and kurtosis for raw SFOR, guided focusing on quantitative relations, and residualized SFOR indicate no violations of normality in any of these measures in any grade level (Skewness < |1.60|; Kurtosis < |1.55|). Separate ANOVAs were run to examine grade level differences in raw SFOR, guided focusing on quantitative relations, and residualized SFOR. Differences in grade level were found for all three indicators, SFOR: $F(2, 248) = 20.19$, $p < .001$, $\eta_p^2 = .14$; Guided focusing on quantitative relations: $F(2, 248) = 17.94$, $p < .001$, $\eta_p^2 = .13$; Residualized SFOR: $F(2, 248) = 9.28$, $p < .001$, $\eta_p^2 = .07$.

Predicting Rational Number Conceptual Knowledge at Time 4

Table 4 presents the bivariate correlations between grade level, predictor variables, and rational number conceptual knowledge at Time 4. Rational Number Test and Extra Density Items were combined to create a more robust sum measure of rational number conceptual knowledge at Time 4. To do so, standardized scores for the size and density sub-tests were calculated in order to provide equal weight for both aspects of rational number conceptual knowledge. There were significant relations between all variables. Both raw SFOR ($r = .64$, $p < .001$) and guided focusing ($r = .46$, $p < .001$) were also related to rational number conceptual knowledge at Time 4.

-Insert Table 4 about here-

In order to further detail the relations between the predictor and outcome variables multiple direct entry linear regressions analyses were run. Predictor variables were grade level, number line estimation, non-verbal intelligence, arithmetic fluency, and residualized SFOR; the outcome variable was rational number conceptual knowledge at Time 4. The order of entry was based on the outlined predictions and the predictive correlations. Incremental R^2 and standardized beta coefficients are displayed to show both the unique contribution of each predictor and the amount of variance added by each predictor of later rational number conceptual knowledge at the given point of entry into the equation.

-Insert Table 5 about here-

Table 5 describes the standardized beta coefficients and R-squared change for each predictor variable. The regression predicting rational number conceptual knowledge was significant, $F(5, 220) = 53.53$, $p < .001$, and indicates that the predictor variables explained 55% of the variance of rational number conceptual knowledge. Grade level, whole number estimation, non-verbal intelligence, arithmetic fluency, and residualized SFOR all uniquely contributed to rational number conceptual knowledge. Even after controlling for all other predictors, residualized SFOR still had a significant contribution to the prediction of rational number conceptual knowledge (ΔR^2 = 4%).

Residualized SFOR in Relation to the Developmental Trajectories of Rational Number Conceptual Knowledge

In order to more directly investigate how SFOR tendency predicts the actual developmental of rational number conceptual knowledge over this time period, a latent variable growth curve model was estimated of the basic Rational Number Test at all four time points (Table 6) and with all predictor variables included as covariates of both initial knowledge and the growth of knowledge of the four time points.

-Insert Table 6 about here-

The slope (S) of participants' Rational Number Test scores over the four time points were estimated to determine the development, while their initial knowledge was estimated by the intercept (I) of the Rational Number Test at Time 1. Grade level, whole number estimation, non-verbal intelligence, arithmetic fluency, and residualized SFOR were included as predictors of the intercept and slope of the Rational Number Test (See Figure 3 for estimated model). The relation between the intercept and slope of rational number conceptual knowledge was also included in the model. Fit indices for the estimated model were sufficient, $\chi^2(15) = 20.60$, $p = .15$, RMSEA = .04, CFI = 1.00, SRMR = .03.

-Insert Figure 3 about here-

As the model reveals, residualized SFOR and non-verbal intelligence were found to be positively related to both the intercept and slope of the Rational Number Test, while the prior rational number conceptual knowledge, grade level, arithmetic fluency, and whole number estimation were only related to the initial knowledge of rational numbers. Given the strong fit of the model, the linear function was considered adequate for modelling students' growth and no other non-linear models were considered.

Discussion

The results of the present study show for the first time that SFOR tendency has a unique impact on rational number conceptual development in late primary school students during the first phase of learning about rational numbers. First, these results demonstrate that there are substantial individual differences in SFOR tendency, and that these differences are not entirely explained by the ability to recognize and describe quantitative relations in the SFOR tasks when explicitly guided to do so. Second, SFOR tendency was found to provide a substantial and unique contribution to the development of rational number conceptual knowledge over two school years of rational number instruction. This contribution remained even after controlling for measures of prior rational number conceptual knowledge, grade level, whole number estimation, non-verbal intelligence, arithmetic fluency, and the ability to recognize and describe quantitative relations. Those students with a higher SFOR tendency are thus shown to have a long-term advantage in the development of rational number conceptual knowledge.

SFOR Tendency in Late Primary School

Previously, individual differences in SFOR tendency have been identified among early primary school children both in Finland (McMullen et al., 2013) and the US (McMullen et al., 2014) by using imitation tasks. The present study indicates that individual differences in SFOR tendency also appear in late primary school when the tendency is measured by paper and pencil tasks requiring students to describe different sets of pictures in writing or drawing. Grade level differences in SFOR tendency indicate that older students were more likely to see the tasks as containing quantitative relations as relevant features, even after taking into account the ability to recognize and describe quantitative relations in the tasks measuring guided focusing on quantitative relations. Despite this, within grade-level there remained substantial differences in SFOR tendency.

More importantly, the residualized SFOR scores reveal that a substantial portion of variance in SFOR tendency is independent from the ability to recognize and describe quantitative relations in the SFOR tasks. This suggests that SFOR tendency is partially distinct from quantitative relational reasoning and can be described as an indicator of the spontaneous focusing of attention on quantitative relations in situations that are not explicitly mathematical. These findings extend previous research on children's SFON tendency (e.g. Hannula & Lehtinen, 2005) and indicate that the theoretical and methodological approach of examining domain-specific attentional processes is a fruitful pursuit, which can inform on individual differences in the development of the number concept, from the first experiences with whole numbers through the extension of the number concept to include rational numbers.

The two tasks included as a measure of SFOR tendency utilized different representations of quantitative relations. Despite this, they were highly reliable as a combined measure of SFOR tendency. The results of the present study indicate that younger students have a lower SFOR tendency. However, only a small number of students were unable to provide an adequate description of the quantitative relations on either the SFOR or guided focusing tasks, suggesting that the tasks were appropriate for the students' age and ability. As well, given previous evidence of SFOR tendency in early primary school (McMullen et al., 2013, 2014), the lower frequency of SFOR responses in grade three and four students suggests that the types of quantitative relations used in SFOR tasks has an impact on inter-individual variability in responses.

In the teleportation task, a SFOR score required focusing on the multiplicative relations between the two collections of three sets of objects and explicitly describing that relation or replicating it in a drawing. In the plate task, the quantitative relations required for a SFOR response involved part-whole relations between the different types of food and the whole plate of food. While these relations were represented as continuous quantities, exact numerical relations were required in order to describe the quantitative relations. The use of the common term, SFOR, to describe focusing on fundamentally different quantitative relations (both within this study and in previous studies of SFOR) is necessary in order to not heedlessly limit the investigation to only specific quantitative relations. There are a number of commonalities across these different instances of SFOR that form a coherent framework for delineating the phenomenon. In order to use quantitative relations to solve SFOR tasks, including the Teleportation and Plate tasks used in the present study, participants must, 1) define a whole set or sub-set of objects or quantity (e.g. a whole plate of food, a set of milk cartons), 2) define (at least) a second set or sub-set, 3) identify a quantitative relation between these two or more (sub-)sets, and 4) explicitly use or describe this quantitative relation in their response.

While these features were also found in the Road task, the low correlation of the Road task and other SFOR tasks indicated it should not be included as a part of the raw SFOR variable. This may be a consequence of the collusion of other aspects of the task interfering with the need to respond relationally. For example, it was possible to quite accurately describe the location of the vehicles based on geographic location, limiting the need to mention the portion of the journey that had been completed in relational terms.

SFOR Tendency's Contribution to Rational Number Development

Continuity and conflict in the development of number concepts from natural to rational numbers is currently facing valuable scrutiny within educational and psychological research (Siegler, Fazio, Bailey, & Zhou, 2013; Vamvakoussi, 2015). Recent studies indicate that fraction magnitude and arithmetic knowledge may be predicted by early whole number competences when looking at components of rational numbers that are both consistent and inconsistent with natural numbers (Bailey et al., 2014; Jordan et al., 2013; Vukovic et al., 2014). The results of the present study indicate that when considering those aspects of rational numbers that are incongruent with natural number features, SFOR tendency uniquely contributes to the development of rational number conceptual knowledge. It has been suggested that the natural number bias plays an important role in difficulties with rational number conceptual knowledge (Ni & Zhou, 2005). In particular, students often over-apply natural number features when reasoning about the size and density of rational numbers (McMullen et al., 2015b; Van Hoof et al., 2015).

The present study indicates that those students with a higher SFOR tendency may be more successful in overcoming the natural number bias when reasoning about rational numbers. Previously, both SFON tendency before school age and SFOR tendency in early primary school have been shown to be related to later rational number conceptual knowledge (McMullen et al., 2014, 2015a). The present study provides even stronger evidence that SFOR tendency plays an important role in the transition from reasoning about numbers as having features of natural numbers to a mathematically adequate understanding of the concepts of rational numbers. It is possible that the extra self-initiated practice with quantitative relational reasoning afforded by a higher SFOR tendency may strongly benefit the development of rational number conceptual knowledge, especially in providing more opportunities to identify the shortcomings of whole numbers in explaining quantities in many everyday situations (McMullen et al., 2015a).

Limitations and Future Directions

There are many limitations to the present study that should be addressed in future studies of SFOR tendency and its relation to rational number development. Future studies could include more valid measures of number line estimation (e.g. Siegler et al., 2011). Using a larger number of these items would also allow for the measurement of absolute error offering more variance in individuals' responses (e.g. Bailey et al., 2014). Unlike other control variables which were measured at Time 1, we were only able to measure whole number line estimation at Time 2, due to time constraints by the schools. Even though this is not optimal for the growth curve analyses of the study, it is unlikely that this would have jeopardized the conclusions of the study. Based the expected stability of whole number estimation in this age group (Ebersbach, Luwel, Frick, Onghena, & Verschaffel, 2008), it is unlikely that three months' time difference in the beginning of the follow-up would make a substantial difference in the relationship between SFOR, number line estimation skills and rational number knowledge.

As well, including a measure of fraction calculation that covers the knowledge of rational number operations would also broaden the scope of the rational number test (Van Hoof et al., 2015). Most importantly, the measure of guided focusing on quantitative relations in the present study is only based on two items, a more robust measure, possibly using all items from the corresponding SFOR tasks (e.g. McMullen et al., 2014) should be included in future studies.

The main challenge, however, is to develop a deeper theoretical description of the SFOR concept and, subsequently, develop more robust methods for measuring it. One issue regarding the measurement of SFOR tendency that has yet to be addressed is the assessment of SFOR tendency at multiple time points. Observing the stability of SFOR tendency across multiple time points, along with the possible reciprocal effects of SFOR tendency and rational number conceptual knowledge are necessary for a better understanding of how the two are inter-related. It would be interesting to see whether a similar pattern of reciprocal development would appear between SFOR and rational number conceptual knowledge at school age as there seems to be in between SFON and verbal counting skills before school age (Hannula & Lehtinen, 2005). In the current study, it was not investigated how spontaneous focusing on the different types of quantitative and numerical relations are related. Future studies must specifically address how SFOR tendency differs between different types of quantitative relations, especially along the developmental continuum (cf. Degrande, Verschaffel, & Van Dooren, 2015). Focusing attention on different types of quantitative relations may have different roles in the development of mathematical skills throughout schooling. These quantitative relations include exact and approximate relations that are additive, multiplicative, part-whole, or proportional in nature.

While the present study was able to confirm that SFOR tendency's relation to later rational number conceptual knowledge was not explained by grade level, other mathematical skills, including the guided recognition and description of quantitative relations, or general intelligence, more evidence is still needed to better understand the exact nature of individual differences in SFOR tendency as a unique aspect of mathematical development. It has been argued that a strong SFOR tendency supports the learning of rational numbers by contributing to more self-initiated practice with quantitative and numerical relations in a variety of everyday situations (McMullen et al., 2014). However in this study SFOR tendency was measured only in special test situations with only a few SFOR tasks, and empirical evidence of the relation between SFOR tendency in testing situations and SFOR in everyday situations remains a necessity. As well, examining the role of SFOR tendency in the development of knowledge of other mathematical topics would help refine the understanding of the contribution of SFOR to the development of mathematical skills in general. Finally, other non-ability-related features of mathematical development, for example motivational or affective dispositions, should be considered in future studies of SFOR tendency (e.g. Edens & Potter, 2013).

Conclusions and Educational Considerations

A great deal of learning and development occurs outside of formal classroom situations (Bransford et al., 2006). It is short sighted and naïve to believe this only applies to language-based domains. Skills and competences with mathematics in informal environments are not automatically applied to formal math learning (Nunes, Schliemann, & Carraher, 1993). Despite this, researchers and educators often treat the investigation and measurement of mathematical competences and development as something that is only done in highly stylized and mathematically obvious situations. Investigations of spontaneous quantitative focusing tendencies suggest that expanding the scope of the mathematical development research to also include situations which are not explicitly mathematical is needed (for reviews, see Batchelor, 2014; Hannula, 2005; McMullen, 2014).

While most students struggle with learning about rational numbers, especially those aspects which are incongruent with rational numbers, the present study suggests that there are some who have leg up in this process. Those students who are more likely to spontaneously focus their attention on quantitative relations, when not guided to do so, are also more likely to have more well-developed rational number conceptual knowledge in late primary school. These same students are likewise more likely to gain more rational number conceptual knowledge over a year and a half period than their peers.

It is certain that more evidence based in intervention studies is needed in order to fully understand the educational implications of the relation between SFOR tendency and rational number development. However, the results of the present study, and previous studies of spontaneous quantitative focusing tendencies, make it clear that educators need to approach mathematics teaching and learning in a way that expands the scope of mathematical practice and relevance beyond the typical highly regulated situations presented in classrooms and mathematics textbooks. The “messy” mathematics that can be found throughout everyday life provides rich opportunities to not only practice mathematical competences and knowledge, but also experience the limitations of prior concepts of, for example, the number system (McMullen et al., 2015a). These experiences may prove invaluable to coming to fully understand the nature of the mathematically correct number system, and support the tricky transition to a mathematically correct understanding of rational numbers.

References

Bailey, D., Siegler, R. S., & Geary, D. C. (2014). Early predictors of middle school fraction knowledge. *Developmental Science, 17*, 775–785. doi:10.1111/desc.12155

Batchelor, S. (2014). *Dispositional Factors Affecting Children's Early Numerical Development*. Unpublished doctoral thesis. Loughborough University. Retrieved from: https://dspace.lboro.ac.uk/2134/17474

Bempeni, M., & Vamvakoussi, X. (2015). Individual differences in students' knowing and learning about fractions: Evidence from an in-depth qualitative study. *Frontline Learning Research, 3*, 17–34. doi:10.14786/flr.v3i1.132

Boyer, T. W., Levine, S. C., & Huttenlocher, J. (2008). Development of proportional reasoning: where young children go wrong. *Developmental Psychology, 44*, 1478–1490. doi:10.1037/a0013110

Bransford, J. D., Barron, B., Pea, R., Meltzoff, A., Kuhl, P., Bell, P., & Sabelli, N. (2006). Foundations and opportunities for an interdisciplinary science of learning. In K. Sawyer (Ed.), *The Cambridge Handbook of the Learning Sciences* (pp. 19–34). New York: Cambridge University Press.

Byrne, B. M. (2012). *Structural equation modeling with Mplus : basic concepts, applications, and programming*. New York: Routledge.

Confrey, J., Maloney, A., Nguyen, K., Mojica, G., & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (pp. 1–8). Thessaloniki, Greece: PME.

Degrande, T., Verschaffel, L., & Van Dooren, W. (2015). Reconsidering SFOR: Characterizing children's Spontaneous Focus On quantitative Relations. Manuscript submitted for publication.

Ebersbach, M., Luwel, K., Frick, A., Onghena, P., & Verschaffel, L. (2008). The relationship between the shape of the mental number line and familiarity with numbers in 5- to 9-year old children:

Evidence for a segmented linear model. *Journal of Experimental Child Psychology, 99*, 1–17. doi:10.1016/j.jecp.2007.08.006

Edens, K. M., & Potter, E. F. (2013). An exploratory look at the relationships among math skills, motivational factors and activity choice. *Early Childhood Education Journal, 41*, 235–243. doi:10.1007/s10643-012-0540-y

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Hamlett, C. L., Cirino, P. T., … Changas, P. (2013). Improving at-risk learners' understanding of fractions. *Journal of Educational Psychology, 105*, 683–700. doi:10.1037/a0032446

Hallett, D., Nunes, T., & Bryant, P. (2010). Individual differences in conceptual and procedural knowledge when learning fractions. *Journal of Educational Psychology, 102*, 395–406. doi:10.1037/a0017486

Hannula, M. M. (2005). *Spontaneous focusing on numerosity in the development of early mathematical skills*. Turku, Finland: Painosalama.

Hannula, M. M., & Lehtinen, E. (2005). Spontaneous focusing on numerosity and mathematical skills of young children. *Learning and Instruction*, *15,* 237–256. doi:10.1016/j.learninstruc.2005.04.005

Hannula, M. M., Lepola, J., & Lehtinen, E. (2010). Spontaneous focusing on numerosity as a domain-specific predictor of arithmetical skills. *Journal of Experimental Child Psychology*, *107*, 394–406. doi:10.1016/j.jecp.2010.06.004

Hannula-Sormunen, M. M. (2014). Spontaneous focusing on numerosity and its relation to counting and arithmetic. In R. Cohen Kadosh & A. Dowker (Eds.), *Oxford Handbook of Mathematical Cognition.* Oxford: Oxford University Press. doi:10.1093/oxfordhb/9780199642342.013.018

Hannula-Sormunen, M. M., Lehtinen, E., & Räsänen, P. (2015). Children's preschool subitizing, spontaneous focusing on numerosity and counting skills as predictors of mathematical performance 6-7 years later at school. *Mathematical Thinking and Learning*. Advance online publication. doi:10.1080/10986065.2015.1016814

Hecht, S. A., & Vagi, K. J. (2012). Patterns of strengths and weaknesses in children's knowledge about fractions. *Journal of Experimental Child Psychology, 111*, 212–229. doi:10.1016/j.jecp.2011.08.012

Hu, L., & Bentler, P. M. (1999). Cutoff criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. *Structural Equation Modeling: A Multidisciplinary Journal, 6*, 1–55. doi:10.1080/10705519909540118

Jordan, N., Hansen, N., Fuchs, L., Siegler, R. S., Gersten, R., & Micklos, D. (2013). Developmental predictors of fraction concepts and procedures. *Journal of Experimental Child Psychology*, *116*, 45–58. doi:10.1016/j.jecp.2013.02.001

Kucian, K., Kohn, J., Hannula-Sormunen, M. M., Richtmann, V., Grond, U., Käser, T., & von Aster, M. (2012). Kinder mit Dyskalkulie fokussieren spontan weniger auf Anzahligkeit [Children with Developmental Dyscalculia Focus Spontaneously Less on Numerosities]. *Lernen Und Lernstörungen, 1*, 241–253. doi:10.1024/2235-0977/a000024

Martinie, S. (2007). *Middle School Rational Number Knowledge*. Doctoral Disseration: Kansas State University.

McMullen, J. (2014). *Spontaneous Focusing on Quantitative Relations and the Development of Rational Number Conceptual Knowledge*. Turku, Finland: Painosalama.

McMullen, J., Hannula-Sormunen, M. M., & Lehtinen, E. (2011). Young children's spontaneous focusing on quantitative aspects and their verbalizations of their quantitative reasoning. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (pp. 217–224). Ankara, Turkey: PME.

McMullen, J., Hannula-Sormunen, M. M., & Lehtinen, E. (2013). Young children's recognition of quantitative relations in mathematically unspecified settings. *Journal of Mathematical Behavior, 32*, 450–460. doi:10.1016/j.jmathb.2013.06.001

McMullen, J., Hannula-Sormunen, M. M., & Lehtinen, E. (2014). Spontaneous focusing on quantitative relations in the development of children's fraction knowledge. *Cognition and Instruction*, *32*, 198–218. doi:10.1080/07370008.2014.887085

McMullen, J., Hannula-Sormunen, M. M., & Lehtinen, E. (2015a). Preschool spontaneous focusing on numerosity predicts rational number knowledge six years later. *ZDM - Mathematics Education*, *47*. Advance Online Publication. doi:10.1007/s11858-015-0669-4

McMullen, J., Laakkonen, E., Hannula-Sormunen, M., & Lehtinen, E. (2015b). Modeling the developmental trajectories of rational number concept(s). *Learning and Instruction, 37*, 14–20. doi:10.1016/j.learninstruc.2013.12.004

Merenluoto, K., & Lehtinen, E. (2004). Number concept and conceptual change: towards a systemic model of the processes of change. *Learning and Instruction*, *14*, 519–534. doi:10.1016/j.learninstruc.2004.06.016

Möhring, W., Newcombe, N. S., Levine, S. C., & Frick, A. (2015). Spatial Proportional Reasoning is Associated with Formal Knowledge About Fractions. *Journal of Cognition and Development.* Advance Online Publication. doi:10.1080/15248372.2014.996289

Muthén, L. K., & Muthén, B. O. (2008-2012). *Mplus user's guide*. *Seventh Edition*. Los Angeles, CA: Muthén and Muthén.

Ni, Y., & Zhou, Y.-D. Y. (2005). Teaching and Learning Fraction and Rational Numbers: The Origins and Implications of Whole Number Bias. *Educational Psychologist*, *40*, 27–52. doi:10.1207/s15326985ep4001_3

Nunes, T., Schliemann, A. D., & Carraher, D. W. (1993). *Street mathematics and school mathematics.* Cambridge University Press.

Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction*, *28,* 64–72. doi:10.1016/j.learninstruc.2013.05.003

Potter, E. F. (2009). Spontaneous focusing on numerosity: Motivational and skill correlates in young children in a public preschool and kindergarten program. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the 31st annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 152–155). Atlanta: Georgia State University Press.

Prieto, L., Alonso, J., & Lamarca, R. (2003). Classical test theory versus Rasch analysis for quality of life questionnaire reduction. *Health and Quality of Life Outcomes*, *27*. doi:10.1186/1477-7525-1-27

Raven, J. C. (1976). *Coloured progressive matrices*. Oxford: Psychologist Press.

Reyna, V. F., & Brainerd, C. J. (2007). The importance of mathematics in health and human judgment: Numeracy, risk communication, and medical decision making. *Learning and Individual Differences, 17*, 147–159. doi:10.1016/j.lindif.2007.03.010

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing Conceptual Understanding and Procedural Skills in Mathematics: An Iterative Process. *Journal of Educational Psychology, 93*, 346–362. doi: 10.1037/0022-0663.93.2.346

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., … Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*, 691–697. doi:10.1177/0956797612440101

Siegler, R. S., Fazio, L., Bailey, D., & Zhou, X. (2013). Fractions: the new frontier for theories of numerical development. *Trends in Cognitive Sciences*, *17,* 13–19. doi:10.1016/j.tics.2012.11.004.

Siegler, R. S., Thompson, C., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology, 62,* 273–296. doi:10.1016/j.cogpsych.2011.03.001

Singer-Freeman, K., & Goswami, U. (2001). Does half a pizza equal half a box of chocolates? Proportional matching in an analogy task. *Cognitive Development, 16*, 811–829. doi:10.1016/S0885-2014(01)00066-1

Sophian, C. (2000). Perceptions of proportionality in young children: matching spatial ratios. *Cognition, 75*, 145–70. doi:10.1016/S0010-0277(00)00062-7

Spinillo, A. G., & Bryant, P. E. (1999). Proportional Reasoning in Young Children : Part – Part Comparisons about Continuous and Discontinous Quantity. *Mathematical Cognition, 5*, 181–197. doi:10.1080/135467999387298

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, *14*, 503–518. doi:10.1016/j.learninstruc.2004.06.015

Vamvakoussi, X. (2015). The development of rational number knowledge: Old topic, new insights. *Learning and Instruction*, *37,* 50–55. doi:10.1016/j.learninstruc.2015.01.002

Vamvakoussi, X., Christou, K. P., Mertens, L., & Van Dooren, W. (2011). What fills the gap between discrete and dense? Greek and Flemish students' understanding of density. *Learning and Instruction, 21,* 676–685. doi:10.1016/j.learninstruc.2011.03.005

Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. *The Journal of Mathematical Behavior, 31*, 344–355. doi:10.1016/j.jmathb.2012.02.001

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: a conceptual change approach. *Learning and Instruction, 14*, 453–467. doi:10.1016/j.learninstruc.2004.06.013

Vamvakoussi, X., & Vosniadou, S. (2010). How Many Decimals Are There Between Two Fractions ? Aspects of Secondary School Students' Understanding of Rational Numbers and Their Notation. *Cognition and Instruction, 28,* 181–209. doi:10.1080/07370001003676603

Van Dooren, W., Lehtinen, E., & Verschaffel, L. (2015). Unraveling the gap between natural and rational numbers. *Learning and Instruction, 37,* 1–4. doi:10.1016/j.learninstruc.2015.01.001

Van Hoof, J. (2015). *Get rational: An investigation of the role of learners' natural number knowledge in rational number tasks*. Doctoral Disseration, KU Leuven.

Van Hoof, J., Janssen, R., Verschaffel, L., & Van Dooren, W. (2015). Inhibiting natural number knowledge in fourth graders: towards a comprehensive test instrument. *ZDM - Mathematics Education*, *47*. Advance Online Publication. doi:10.1007/s11858-014-0650-7

Van Hoof, J., Lijnen, T., Verschaffel, L., & Van Dooren, W. (2013). Are secondary school students still hampered by the natural number bias? A reaction time study on fraction comparison tasks. *Research In Mathematics Education*, *15*, 154–164. doi:10.1080/14794802.2013.797747

Vosniadou, S. (2014). Examining cognitive development from a conceptual change point of view: The framework theory approach. *European Journal of Developmental Psychology, 11,* 645-661.. doi:10.1080/17405629.2014.921153

Vosniadou, S., & Verschaffel, L. (2004). Extending the conceptual change approach to mathematics learning and teaching. *Learning and Instruction*, *14*, 445–451. doi:10.1016/j.learninstruc.2004.06.014

Vukovic, R. K., Fuchs, L. S., Geary, D. C., Jordan, N. C., Gersten, R., & Siegler, R. S. (2014). Sources of individual differences in children's understanding of fractions. *Child Development, 85*, 1461–1476. doi:10.1111/cdev.12218

Woodcock, R. W., McGrew, K. S., Mather, N. (2001). *Woodcock–Johnson III Tests of Achievement*. Itasca, IL: Riverside Publishing.

Figures and Tables

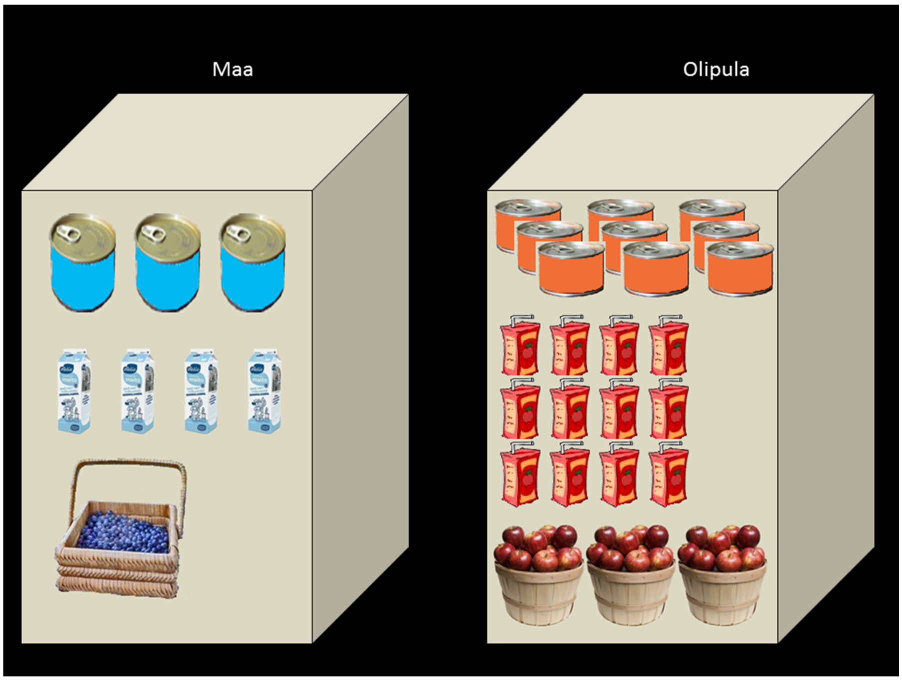

Figure 1. An example of objects used in the teleportation tasks. Original material is on the left and 'teleported' material on the right.

Figure 2. Example trial from the Plate task, also used in guided focusing on quantitative relations task.

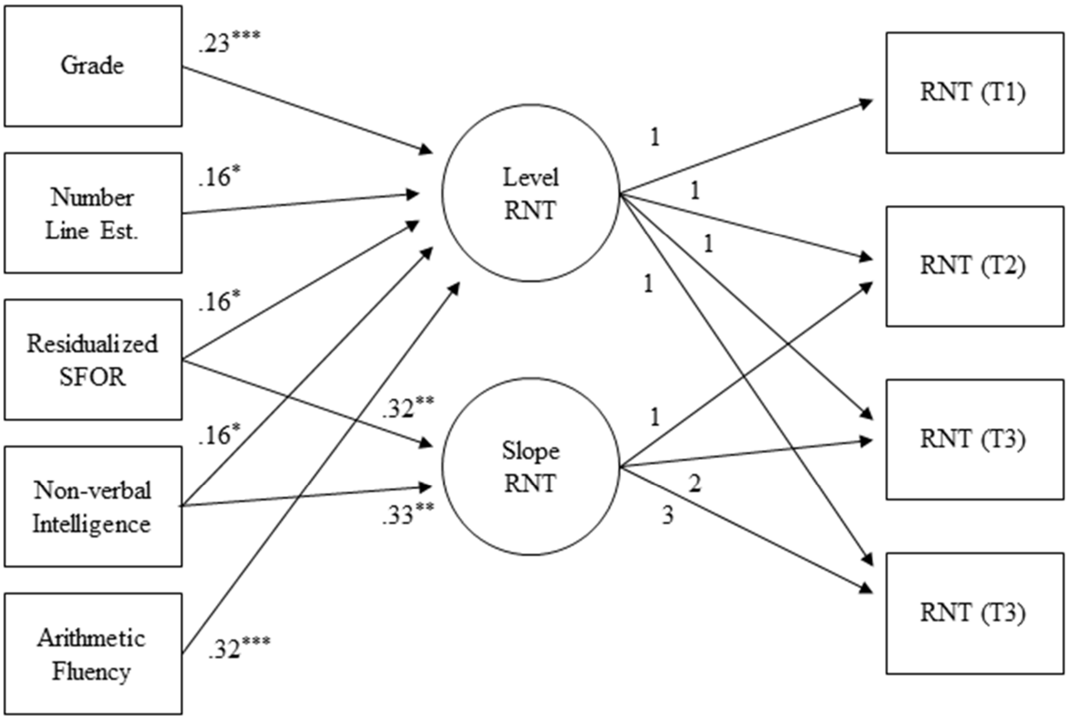

Figure 3. Latent growth curve model of rational number knowledge from Time 1 to Time 4, with predictor variables (Grade Level, Number Line Estimation, Residualized SFOR, Non-Verbal Intelligence, and Arithmetic Fluency).. Level RNT = intercept of Rational Number Test (i.e. initial knowledge), Slope RNT = linear slope of Rational Number Test over Times 1, 2, 3, and 4 (i.e. knowledge growth). Only significant standardized regression coefficients are presented. $^{*}p < .05$, $^{**}p < .01$, $^{***}p < .001$.

Table 1

Descriptive Statistics for Predictor and Outcome Variables (n = 251)

	Mean	Std. Dev.	Skewness	Kurtosis	Range
Predictors – Time 1					
Raw SFOR	3.84	3.84	1.18	0.93	0-16
Teleportation	2.51	2.44	0.83	-0.29	0-8
Plate	1.33	2.21	1.67	1.78	0-8
Guided focusing on quantitative relations	1.44	0.74	-0.90	-0.60	0-2
Predictors – Time 2					
Number line estimation	-.08	.07	-2.04	4.52	-.46– -.01
Non-verbal IQ	14.01	3.54	-0.84	0.39	2-20
Arithmetic Fluency	59.87	18.14	0.48	0.98	17-123
Outcomes – Time 4					
Size Items	9.76	4.74	-0.12	-1.39	0-16
Density Items	5.76	5.29	1.91	3.31	0-24

Table 2

Means and standard deviations (in parentheses) of variables by Grade Level.

	Grade 3 (n = 84)	Grade 4 (n = 75)	Grade 5 (n = 92)
Number line estimation	-.09	-.08	-.06
	(.07)	(.08)	(.07)
Non-verbal IQ	12.95	14.25	14.77
	(3.43)	(3.41)	(3.53)
Arithmetic Fluency	49.31	63.26	66.79
	(15.17)	(18.42)	(16.06)
Rational Number Tests			
Time 1	3.02	7.68	10.70
	(3.10)	(4.93)	(5.91)
Time 2	3.95	9.4	12.02
	(3.29)	(4.62)	(6.18)
Time 3	4.08	10.37	13.28
	(3.61)	(5.67)	(6.41)
Time 4	6.62	12.29	15.01
	(4.38)	(5.86)	(6.21)
Size Items	6.22	10.59	12.23
	(3.98)	(4.15)	(3.91)
Density Items	3.21	6.16	7.70
	(1.92)	(5.38)	(6.29)

Table 3

Means (and Standard Deviations) of Raw SFOR, Guided focusing on quantitative relations, and Residualized SFOR scores at Time 1 by Grade Level.

Grade Level	Raw SFOR	Guided focusing on quantitative relations	Residualized SFOR
3 (n = 84)	1.99	1.07	-0.30
	(2.21)	(0.83)	(0.63)
4 (n = 75)	3.97	1.59	-0.05
	(3.28)	(0.59)	(0.91)
5 (n = 92)	5.41	1.65	0.32
	(4.66)	(0.62)	(1.23)

Table 4

Correlations among Predictor and Outcome Variables

	1	2	3	4	5
1. Grade Level					
2. Residualized SFOR	.26***				
3. Number Line Estimation	.19**	.22***			
4. Non-Verbal IQ	.21***	.35***	.22***		
5. Arithmetic Fluency	.40***	.34***	.26***	.33***	
6.Rational Number Conceptual Knowledge (T4)	.50***	.50***	.37***	.54***	.51***

Note. ** $p < .01$, *** $p < .001$

Table 5

Hierarchical Regression Analyses: Effects of residualized SFOR and Other Factors on Rational Number Conceptual Knowledge at Time 4.

Variable	β (S.E.)	95% CI	$(\Delta R^2)^a$
Grade Level	$.27^{***}$ (.05)	[.17, .37]	$.25^{***}$
Whole Number Line Estimation	$.16^{**}$ (.05)	[.06, .25]	$.08^{***}$
Non-verbal intelligence	$.29^{***}$ (.05)	[.20, .40]	$.14^{***}$
Arithmetic Fluency	$.19^{***}$ (.05)	[.08, .30]	$.04^{***}$
Residualized SFOR	$.22^{***}$ (.05)	[.11, .31]	.04***
Total R^2			.55

Note. $^{**}p < .01$, $^{***}p < .001$

$^a R^2$ change after all other variables have been entered

Table 6

Descriptive statistics of Rational Number Test at all time-points

	Mean	St. Dev.	Skew	Kurt.	Range
Time 1	7.23	5.79	0.76	-0.38	0-24
Time 2	8.54	5.97	0.72	-0.16	0-28
Time 3	9.32	6.61	0.47	-0.67	0-28
Time 4	11.44	6.57	2.44	-0.65	0-28