Modeling the developmental trajectories of rational number concept(s)

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Abstract

The present study focuses on the development of two sub-concepts necessary for a complete mathematical understanding of rational numbers, a) representations of the magnitudes of rational numbers and b) the density of rational numbers. While difficulties with rational number concepts have been seen in students' of all ages, including educated adults, little is known about the developmental trajectories of the separate sub-concepts. We measured 10- to 12-year-old students' conceptual knowledge of rational numbers at three time points over a one-year period and estimated models of their conceptual knowledge using latent variable mixture models. Knowledge of magnitude representations is necessary, but not sufficient, for knowledge of density concepts. A Latent Transition Analysis indicated that few students displayed sustained understanding of rational numbers, particularly concepts of density. Results confirm difficulties with rational number conceptual change and suggest that latent variable mixture models can be useful in documenting these processes.

1 Introduction

Students have serious difficulties with learning about fractions and decimals, and many educated adults do not have an adequate conceptual understanding of rational numbers (Merenluoto & Lehtinen, 2004; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Van Hoof, Vandewalle, Verschaffel, & Van Dooren, this issue; Vosniadou & Verschaffel, 2004). Despite having many features that can be generalized from natural numbers (Steffe & Olive, 2010; Torbeyns, Schneider, Xin, & Siegler, this issue), learning and understanding rational number concepts has proven to be challenging for most learners. One issue with learning about rational numbers is that there are many features of natural numbers that cannot be extended to rational numbers. Thus, learning about rational numbers is not a matter of simply developing a deeper or broader understanding of natural numbers. Understanding rational numbers requires significant change in reasoning about number concepts, for example: A rational number with smaller integers can represent a larger magnitude (e.g. in the denominator of a fraction); there are always an infinite number of numbers between any two numbers; one cannot say what is the next number in a sequence of rational numbers. All of these concepts of rational numbers clash with students' previous understandings of the nature of numbers. Thus, deep and significant conceptual change is required in order to fully grasp the nature of rational numbers, and this change is hard to come by.

The present study tracks, over a one-year period, the development of conceptual knowledge of third to fifth grade children. We aim to identify key stages in the development towards a coherent and mathematically correct conceptual understanding of rational numbers in students who are first learning about rational numbers. In doing so, we hope that what emerges is a more fully developed understanding of rational number conceptual change processes. In particular, we aim to uncover how the different sub-concepts of rational numbers develop.

1.1 Rational number conceptual change

Conceptual change with rational numbers is a complex phenomenon, which can be operationalized with different sub-concepts. In this study, we focus on concepts related to the representations of the magnitudes of fractions and/or decimals and the density of fractions or decimals. Learning both of these sub-concepts violates the previous concepts students have of numbers, based on their natural number knowledge (DeWolf & Vosniadou, this issue; Vamvakoussi, Christou, Mertens, & Van Dooren, 2011). While difficulties with the different sub-concepts have been demonstrated in several studies, little is known about the developmental trajectories of these sub-concepts.

Natural number magnitudes are easily and directly perceived by symbolic representations and can only be represented by one term. However, rational numbers do not to follow the same rules (DeWolf & Vosniadou, this issue; Merenluoto & Lehtinen, 2004; Ni & Zhou, 2005; Schneider & Siegler, 2010). First, rational numbers can be represented by an infinite number of terms. Not only can the same magnitude be represented in both decimal and fraction form; there are an infinite number of ways to represent a magnitude within both of these forms (e.g. 0.5 = 0.50 = 0.500 = ½ = 2/4 = 3/6 =…). Prior to rational number learning, students can be sure that each term represents a single magnitude, and that each magnitude is represented by a single term. The expansion of numerical representation options represents a fundamental change in the number concept, one that can be difficult for students to traverse.

Second, the magnitudes of rational numbers are not immediately perceived, as is the case with natural numbers. Determining a fraction's magnitude requires the understanding that the magnitude is represented by a ratio between the two terms. Less coherent models of rational numbers may lead students to view fractions as two separate integers. This component based reasoning may lead to the over-use of natural number based reasoning (Merenluoto & Lehtinen, 2002; 2004). As well, decimal numbers do not follow the pattern that more digits indicate a larger magnitude (1.65 is smaller than 1.7), as is the case with natural numbers. Failures with the comparison of decimal numbers may also be due to students' component based comparisons of the terms before and after the decimal (as in 65>7, therefore 1.65>1.7) (see Durkin & Rittle-Johnson, this issue). Because of these large differences in representation modes, students' prior concepts of natural numbers hinder the development of a mathematical understanding of rational number magnitudes. Instead, substantial conceptual change is needed.

Even more demanding conceptual change is necessary for students to understand the density of rational numbers (Hannula, Pehkonen, Maijala, & Soro, 2006; Merenluoto & Lehtinen, 2002; Vamvakoussi et al, 2011; Vamvakoussi & Vosniadou, 2004; 2010). The two main conceptual differences concerning concepts of density are a) the sequencing of numbers and b) the presence of a successor number. Learning the sequence of natural numbers, based on their understanding of successor terms, is one of the first numerical experiences most children have and students conceive of natural numbers as being discrete quantities (Ni & Zhou, 2005; Vosniadou, Vamvakoussi, & Skopeliti, 2008). However, rational numbers do not have a discrete sequence that can be reproduced. Relatedly, while with natural numbers it is always possible to distinguish the successor number of any number, this is untrue when the number concept is expanded to rational numbers. In total, these characteristics of rational and natural numbers lead to a large disparity in their conceptualization. It is possible to determine the terms between any two non-equal natural numbers. However, there are an infinite number of rational numbers in between any two non-equal rational numbers. Coming to understand rational numbers as dense sets without successors is a challenge involving substantial conceptual change in the nature of numbers (Harnett & Gelman, 1998).

For both the comparison and density of rational numbers, previous research has outlined the concepts based on natural number that require fundamental conceptual change. However, little previous research has focused on the developmental stages that students go through in traversing this conceptual change. Adults and university students have been found to possess incorrect concepts about rational numbers indicating that many students do not ever successfully leave behind their natural number based concepts of rational numbers (DeWolf & Vosniadou, this issue; Merenluoto & Lehtinen, 2005).

1.2 Latent Profile and Latent Transition Analyses

Latent Profile Analysis (LPA) and Latent Transition Analysis (LTA) are employed in present study in the hopes that they can provide new insight into the developmental trajectories related to rational number conceptual change. Latent variable mixture models provide two main advantages over traditional modeling, such as cluster analysis, for the modeling of a number of social and behavioral characteristics (Muthén, 2001; Nylund, Asparouhov, & Muthén, 2007). First, latent variable mixture models allow for the testing of the statistical fit of different classifications (e.g. 3 vs. 4 classes). Thus, latent variable mixture models could provide much needed added value over traditional cluster analysis in investigating conceptual change. If sufficient theoretical assumptions are used, prior to testing, to determine potential classification models, latent variable mixture modeling can corroborate the most statistically appropriate model with a theoretically sound classification system. In this way, latent variable mixture models may help confirm whether there are separate developmental stages, and what these stages look like, throughout the process of conceptual change with rational numbers.

Second, Latent Transition Analyses may be useful for detailing complex trajectories of conceptual change with rational numbers, as a result of the ability to determine what types of responses appear in different developmental stages. Latent variable mixture models have been shown to be useful for the classification of children’s mental models of the earth (Straatmeier et al., 2008) and in analyzing the development of children’s concepts of sinking and floating (Schneider & Hardy, 2013). LTA is able to assess the probability of subjects changing class membership across two or more time-points, which may be useful as a method to detail the multi-faceted nature of conceptual change (Lanza & Collins, 2008; Nylund, 2007). Thus, Latent Transition Analysis may help determine how students transition from early misconceptions of rational numbers to successful rational number conceptual change.

1.3 Research Questions and Hypotheses

We therefore report on a one-year longitudinal study of third- and fifth-grade students’ conceptual knowledge of rational numbers. We administered a test of rational number knowledge at three time points, and use students’ responses on these tests to create LPA and LTA models. We aim to answer the question of whether there can be found a distinct developmental pattern showing growth of conceptual knowledge of rational numbers that can explained by the conceptual change processes identified above. We expect students to display difficulty in reasoning about concepts of rational numbers, particularly concepts of density (Hypothesis 1a). We expect that it is possible to model different profiles of conceptual knowledge of magnitude representations and density of rational numbers that can be explained by theories of conceptual change with rational numbers (Hypothesis 1b). Developmentally, we expect limited growth of conceptual knowledge of rational numbers (Hypothesis 2a). Particular, we expect conceptual knowledge of density to be less well-developed than conceptual knowledge of magnitude representations (Hypothesis 2b).

2 Methods

2.1 Participants

In total, 263 students (141 female) from two primary schools in Southwest Finland participated in the study. Students were between the ages of 10-12 years (Grades 3-5) at the start of testing. Students were tested at the beginning and at the end of spring term in 2012, and at the beginning of the spring term in 2013. 12 students did not complete testing after the first time point, and an additional 30 had changed schools and were unavailable to complete the study after the second round of testing. Those 42 students who did not complete all time points did not differ from the rest of the sample on initial testing. The sample represented a Finnish urban population, including students from lower-middle class to middle class backgrounds and from diverse ethnic backgrounds.

2.2 Design

Students completed the Rational Number Test (RNT) a total of 3 times over the course of two school years, at the beginning and end of the spring semester, and again at the beginning of the next years' spring term. Data collections coincided with either the start or the finish of participants' regular course on fractions and decimals. Students' regular fraction and decimal courses covered basic topics such as learning about the representations of rational numbers, including the representations of magnitude for both fractions and decimals, and basic arithmetic computation with fractions and decimals. There was no direct instruction on the concepts of density during the basic courses for any grade level covered in the present study. However, students did learn about the base-10 system in the construction of decimal numbers and had exercises involving number line estimation, which indirectly involved some concepts of density, especially that there can always be decimals or fractions between given numbers on the number line. Students also learnt that fractions can always be further broken down when, for example, dividing "pizza" models in smaller units. Time 1 was in February 2012 before their rational number courses, Time 2 was in May 2012 after their rational number courses, and Time 3 was in January 2013 before their rational number courses. Two trained research assistants presented all tasks in the students' home classroom, in a timed group setting.

2.3 Rational Number Test (RNT)

The Rational Number Test (RNT) consisted of 22 multiple choice and short answer items. Students were given 15 minutes to complete the test. The test consisted of three types of problems measuring students' rational number conceptual knowledge:

Comparison of rational numbers, ordering of rational numbers, and density of rational numbers.

Comparison items were multiple-choice including: 3 items comparing two fractions (e.g. "Circle the larger fraction. If the numbers are equal circle both.": 5/8; 4/3), 3 items comparing two decimals (e.g. "Circle the larger decimal…": 0.36; 0.5), and 4 items comparing fractions and decimals (e.g. "Circle the large number…": 1/8; 0.8). Each item was scored as correct or incorrect with a maximum score of 10 for the comparison portion of the test.

Ordering items were short answer responses including: 3 items ordering fractions (e.g. "Put the numbers in order from smallest to largest": 6/8; 2/2; 1/3) and 3 items ordering decimals (e.g. "Put the numbers in order from smallest to largest": 6.79; 6.786; 6.4). Each item was scored as correct or incorrect with a maximum score of 6 for the ordering portion of the test.

Density items were short answer responses including: 2 items on fraction density (e.g. "Are there other fractions between 3/5 and 4/5? If there are, then how many?"), 2 items on decimal density (e.g. "Are there other decimal numbers between 0.3 and 0.4? If there are, then how many?"), and 2 items on fraction infinity concepts (e.g. "What is the smallest possible fraction?"). Each item was scored as incorrect (0 points), partially correct (1 point), or correct (2 points). *Correct responses* were those that displayed a mathematically correct concept of the density or the infinite nature of rational numbers, stating that there are an infinite number of numbers between any two rational numbers, or that it's impossible to say how many there are (e.g. "There are an infinite number [of fractions between 3/5 and 4/5.]"). *Partially correct* responses displayed some understanding that there are a very large number of rational numbers in between any two other rational numbers, but did not include notions of infinity (e.g. “10”, “many”, etc.). *Incorrect responses* displayed no understanding of the density of rational numbers, stating that there are no numbers in between the two numbers (e.g. “There are no numbers [in between 0.3 and 0.4.]”). The maximum score for the density portion of the test was 12.

Basic items of the RNT remained the same across all measurement points, although items were slightly altered (e.g. 3/2 and 5/7 became 4/3 and 5/8) for each subsequent testing to lower test-retest effects. Two independent raters scored 25% of the density items at Time 1 and agreed on 96% of them. Thus, the coding scheme was followed for the rest of the participants at all three time points.

2.4 Analysis

All models were estimated using Mplus version 7.0 (Muthén & Muthén, 1998-2012). The estimation method was maximum likelihood with robust standard errors (MLR), which is a full information approach that can handle missing-at-random data. The analyses of Latent Profiles (LPA) and Latent Transitions (LTA) were carried out as mixture and longitudinal mixture models, in which 1000 and 100 random start values were used in the first and second steps of model estimation, respectively, to ensure the validity of the solution (Geiser, 2013). Model fits were evaluated through a combination of a) statistical indicators (Nylund et al., 2007) and b) substantive theory, in order to determine the suitable number of latent classes and best fitting models. Low values for the information criteria AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) indicate a better fit. Entropy values that approach 1 signify more certainty in the classification. Finally, a significant result of the BLRT (Parametric Bootstrapped Likelihood Ratio Test) supports the *k*-class solution in comparison with the *k-1*-class solution, in other words a significant result on this test suggests that use of that particular model is better than the model that has one less class in it.

3 Results

Table 1 details the descriptive statistics for the three sub-tests at all time points. Because of the high reliability values, we continued with the use of the sub-tests of comparison, ordering, and density in further analysis. A surprising finding was that both the comparison and ordering sub-tests were highly consistent despite encompassing both fraction and decimal items. In fact, both sub-tests had high internal consistency on their own and it was therefore determined to keep them separate in the final analysis.

Table 1.
Descriptive Statistics for Rational Number Test at Times 1, 2, and 3.

	M	SD	Skew	Kurt	Range	Reliability (α)
RNT T1 (N=251)						
Comparison	4.72	3.14	0.23	-1.19	0-10	0.86
Ordering	1.73	2.03	0.89	-0.59	0-6	0.85
Density	0.91	1.59	2.44	6.56	0-9	0.59
RNT T2 (N=239)						
Comparison	5.45	3.05	0.11	-1.34	0-10	0.85
Ordering	1.99	2.07	0.71	-0.88	0-6	0.84
Density	1.27	2.09	2.61	8.78	0-13	0.70
RNT T3 (N=221)						
Comparison	5.55	3.38	0.05	-1.27	0-10	0.85
Ordering	2.47	2.30	0.28	-1.51	0-6	0.87
Density	1.31	2.09	2.70	8.58	0-12	0.78

In general, students were most successful on the comparison tasks, though only 39.8% performed above chance on these items at Time 1, 47.8% doing so at Time 2, and 51.2% performing above chance at Time 3. Students had more trouble with both ordering and density items, with limited success on either sub-test even through Time

3. The disparity in comparison and ordering scores may have resulted from the difference in task difficulty due to chance. While for the comparison items participants had a 1/3 chance of selecting a correct answer, the open-ended response type of the ordering tasks led to only a 1/6 chance of getting a correct answer by guessing. Furthermore, the increase in difficulty on the ordering items can be due to participants having to determine the magnitude of three rational numbers as opposed to only two.

3.1 Latent Profile Analysis

Table 2 outlines the fit indices for the 2-, 3-, 4-, and 5-class solutions. Model fit was determined through a combination of a) statistical indicators and b) substantive theory, allowing for the determination of the most suitable number of latent classes and best fitting models.

- - Insert Table 2 about here - -

Minimum AIC and BIC values and BLRT results do not clearly support one model as superior over the others. However, while the minimum BIC value was not found while running these models, the difference in BIC values diminishes when comparing the change from the 3-class to 4-class models (i.e. BIC differences of 51, 26, and 41 at Times 1, 2, and 3 respectively) with the change from the 4-class to 5-class models (i.e. BIC differences of 17 and 15 at Times 1 and 3 respectively; at Time 2 the 5-class model did not converge). This pattern suggests that the 4-class model is a reasonably parsimonious fit for this data (Nylund et al., 2007).

Furthermore, the 4-class model provides a theoretical advantage over the other models, as it is the most parsimonious model which is able to differentiate between successful conceptual change with the magnitude representations of rational numbers (comparison and ordering sub-tests) and conceptual change with density of rational number concepts. Thus, the 4-class model was seen as the most appropriate model, since a) it had consistently strong statistical indicators at all three time points and b) it was well supported by substantive theory (Lanza & Collins, 2008; Nylund et al., 2007).

3.2 Latent Transition Analysis

3.2.1 Characteristics of the latent classes

Based on the determination of the 4-class model as the most suitable for this data, a latent transition analysis was then run in order to investigate the development of participants' conceptual knowledge of rational numbers over the three time points.Class size varies substantially in this model as can be seen in Table 3. At every time point, nearly half the participants are in Class 1, with the smallest class having less than 10% of participants in it.

Table 3. Latent class membership at Times 1, 2, and 3

	Proportion of sample		
	Time 1	Time 2	Time 3
Low	49.4	45.0	45.8
Average	25.9	29.1	20.7
Mag Rep	18.3	19.1	26.3
High	6.3	6.7	7.2

The different classes are assigned labels in Figure 1 based on our interpretations of the statistical outcomes (Schneider & Hardy, 2013). There are clear differences in overall conceptual knowledge between classes. Class 1, named *Low Conceptual Knowledge* (Low), represents the majority of participants and indexes a clear lack of conceptual understanding of rational numbers, with low mean scores for all sub-tests. Class 2, labeled *Average Conceptual Knowledge* (Average), includes students who have some limited success on the Comparison and Ordering sub-tests, though at a low level, averaging only a few correct answers for each of the sub-tests. In Class 3, labeled *High Magnitude Representation Knowledge* (Representation), participants are highly effective with the Comparison and Ordering sub-tests, however, they perform near average on the Density items. Within Class 4, *High Conceptual Knowledge* (High), participants have extremely high scores on the density portion, providing mathematically accurate responses on the density of rational number, and having high Comparison and Ordering scores, similar to the *Magnitude Representation* class participants. Thus, the key distinction between classes 3 and 4 is success on the density portion of the test.

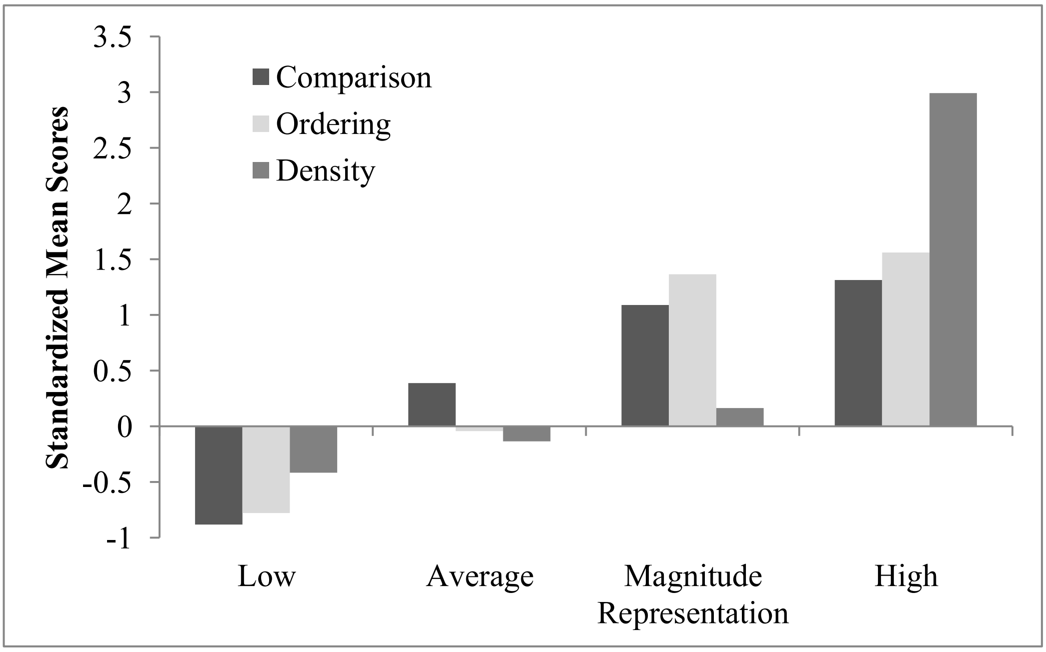

Figure 1. Latent profiles for the four-class model at Times 1, 2, and 3

3.2.2 Latent Transition Probabilities

One result of the LTA is the estimation of Latent Transition Probabilities (LTPs) as parameters of the model. LTPs represent the probability a student would change from a certain class to another or remain in the same class across the different time periods. High probabilities indicate that students are likely to end up in that particular class (be it, remaining in the same class or switching to a different class) across two or more time points. Thus, a latent transition probability can be seen as a coefficient of class stability or instability (Geiser, Lehmann, & Eid, 2006). LTP scores range from 0 to 1, with 1 reflecting that all members of a class are expected to move or remain in the corresponding class across the two time points. The disparities of the frequency of movement between classes allows for a qualitative understanding of how students are developing (or not) their conceptual understanding of rational numbers. A high degree of stability within one category would reflect that students in that category do not change in their conceptual knowledge of rational numbers over time. Whereas, a high LTP between two different categories across two time points (e.g. *Low* in Time 1, and *Average* in Time 2) would reflect that there is a high chance of a student in Low at Time 1 ending up in *Average* at Time 2.

Table 4 details the Latent Transition Probabilities (LTP) from Time 1 to Time 2 and from Time 2 to Time 3. The estimated LTA model closely resembles, in terms of class membership and conceptual knowledge profiles, the 4-class LPA models detailed previously. Therefore, the same class labels will be used when describing the transition analysis.

Table 4. Latent transition probabilities based on 4-class model, probability of class membership from Time 1 to Time 2 and from Time 2 to Time 3.

	Time 2			
Time 1	1	2	3	4
1. Low	**0.91**	0.08	0.01	0.00
2. Average	0.04	**0.93**	0.03	0.00
3. Mag Rep	0.00	0.09	**0.81**	0.10
4. High	0.00	0.00	0.28	**0.72**

	Time 3			
Time 2	1	2	3	4
1. Low	**1.00**	0.00	0.00	0.00
2. Average	0.11	**0.64**	0.23	0.02
3. Mag Rep	0.00	0.03	**0.90**	0.07
4. High	0.00	0.00	0.28	**0.72**

The latent transition probabilities reveal that the *Low* class is the most stable of the four classes. The total probability of remaining in the *Low* class through all three time points is 0.91 (0.91*1.00), revealing that very few participants who began with little understanding of rational number concepts move to more advanced classes during these time periods. In fact, the LTPs reveal that no participants moved from the *Low* class to any other classes between time points 2 and 3, and only a few did so during the first transition period. Overall, 45% of participants remained in the *Low Knowledge* class throughout all time points. The *High* class is the least stable class across both transitions, with participants only having a 51% chance of remaining in the top classification across all three time points (0.72*0.72), having a 28% chance of moving to the *Magnitude Representation* class between each time point. This indicates that the *High* class is fairly unstable and in particular, participants were not very consistent in their conceptual knowledge about density. The lowest stable-class LTP over both transition points was for the *Average* class between time points 2 and 3, with a fairly high number of these participants moving upward to the *Magnitude*

Representation class (LTP=0.23) and some regressing to the *Low* class (LTP=0.11). This indicates that there was potential for conceptual change with comparison and ordering concepts for some of those who had some initial understanding between the later two time points, even if there was little development during the first two time points.

As a whole, there were 64 possible transition paths based on the four-class model over three time points (4^3). Overall, students' conceptual knowledge proved fairly stable, with 82% of participants found to remain in the same classification across all three time points, indicating that only 18% of the participants changed classes at some point. 6% of participants transitioned between the *High* and the *Magnitude Representation* classes in some fashion (i.e. they were classified in the *Magnitude Representation* class and the *High* class during at least one time-point); this was the most common transitional path, though representing only 15 participants. In comparison, only 10 participants remained in the *High* class throughout all time points, again highlighting the inconsistent nature of well-developed conceptual knowledge of rational numbers, in particular concepts of density. The second most common (non-constant) transitional path was movement from the *Average* class to *Magnitude Representation* class by 5% of the participants. This transition mostly came between Time 2 and 3, where 10 participants displayed an increase in conceptual knowledge of magnitude representations of rational numbers. In line with the LTPs presented above, only 3% of participants transitioned from the *Low* to another category, further indicating that there appears little conceptual development with those who have poor initial concepts of rational numbers. The final 4% of transition paths, were made up of transitions to lower levels of conceptual knowledge.

In particular, 4 participants moved from the *Magnitude Representation* class to the *Average* class and 6 participants transitioned from the *Average* to the *Low* class.

4 Discussion

Overall, these results indicate that students have substantial difficulties in understanding rational numbers during their first years of formal learning of rational numbers and only a few students display mathematically mature concepts of either magnitude representations or the density of rational numbers; students had most difficulties with concepts of the density of rational numbers (Hypothesis 1a). These results confirm previous findings that substantial conceptual change is needed for a well-developed understanding of rational numbers (Hypothesis 1b) (Merenluoto & Lehtinen, 2004; Vamvakoussi & Vosniadou, 2004; Vamvakoussi et al, 2011). The use of Latent Profile Analysis revealed that classifying students' conceptual knowledge of rational numbers into four classes was the most appropriate. This allowed for the distinction of those students who had a well-developed understanding of representations of the magnitudes of rational numbers, but did not have well-developed density concepts, from those who had both.

Developmental patterns revealed that few participants displayed substantial conceptual change with either the magnitude representations of rational numbers or the density of rational numbers (Hypothesis 2a). Conceptual change with density concepts was found to be tenuous, as very few participants who exhibited well-developed conceptual knowledge of the density of rational numbers displayed this conceptual understanding consistently throughout the study (Hypothesis 2b). Similar instability of reasoning about density has been found in qualitative case studies (Merenluoto & Lehtinen, 2004).

Latent variable mixture models allow for the investigation of the different developmental trajectories of rational number conceptual knowledge and are able to make patterns of the development of different sub-concepts, magnitude representations and density, visible. Results indicate that in the present sample a 4-class model was the most appropriate for classifying students' response patterns on the rational number test. The major difference between the 4-class model and other models was the separation of those students with the highest scores into two distinct groups: a) those with high scores on the comparison and ordering of rational numbers and b) those with high scores in these sub-tests *and* high scores on the density sub-test. Thus, the 4-class model was able to differentiate between those who showed only a mathematically correct understanding of the magnitude representation of rational numbers from those who displayed mathematically correct concepts of both magnitude representations and the density of rational numbers.

The density of rational numbers has been previously found to be an extremely difficult concept for students, and even adults (e.g. Merenluoto & Lehtinen, 2002, Vamvakoussi & Vosniadou, 2004; Vosniadou & Verschaffel, 2004). The results of the present study reveal that even those with a well-developed understanding of the magnitude representations of rational numbers can fail to grasp the concept of density. In fact, the model in the present study reveals that conceptual knowledge of the magnitude representations of rational numbers is necessary, but not sufficient, for successful conceptual change with density concepts. There was a classification that included students with an understanding of magnitude representations but not density, but there was no class that was made up of students with an understanding of density but not magnitude representations. This may have been a result of the possibility to solve the comparison and ordering portions of the test partially using non-symbolic representations of fractions. However, evidence from the Latent Transition Analysis (LTA) reveals that indeed conceptual knowledge of the density of rational numbers seems to be a fairly unstable trait in the present sample. In particular, only a small number of participants develop an understanding of the density of fractions and decimals and these participants almost exclusive do so only after displaying a well-developed understanding of the magnitude representations of rational numbers.

The results of the LTA reported here confirm that successful conceptual change with rational numbers is rare, hard to come by, and tenuously held. The strength of using LTA models to investigate conceptual change over time is that one can distinguish which classes represent profiles that have response patterns in line with what would be expected by different conceptual models (Straatemeier et al., 2008). In the case of rational numbers, these models would suggest that students have different levels of the conceptual understanding of fractions and decimal numbers. Even after teaching, many students still have difficulties to compare magnitude of fractions and decimal numbers. Even more difficult is to order by magnitude several fractions or decimal numbers. In the model in the present study, The *Low* class, indicating almost complete lack of conceptual understanding of rational numbers, is the biggest and most stable group; very few of these students, who do not have any understanding of the rational number concept in the beginning, grasp it during the one-year follow up period.

The present study suggests limited conceptual change with the magnitude representations of the magnitudes of rational numbers. Around a quarter of the participants in the *Average Conceptual Knowledge* class at Time 2 progressed to the *High Magnitude Representation Knowledge* class at Time 3. This transitional path indicates that some participants who had an initial understanding of the magnitude representations of rational numbers successfully developed a more mathematically correct understanding of the magnitude representations of rational numbers. The contrast between this developmental trend, which occurred between times 2 and 3, and the almost complete lack of movement from the *Average* class to the *Magnitude Representation* class between Times 1 and 2 is particularly surprising given that the explicit teaching of the representations of rational numbers occurred between the first two time points. This suggests that conceptual change with magnitude representations of rational numbers may be a delayed effect of rational number instruction, requiring more time to sink in. More in-depth analysis of the response patterns of those participants who made this transition would shed greater light on this developmental trend.

Even more rarely captured in the LTA model was conceptual change with the density of rational numbers. What was most apparent in the findings was the lack of stability of membership in the *High* class. This indicates that participants were inconsistent in successfully responding to the questions on the density of fractions and decimals in the rational number test. Furthermore, there were very few participants who displayed mathematically appropriate concepts of the density of rational numbers. The few participants who did seem to display successful conceptual change with density concepts almost exclusively did so after having a robust understanding of the magnitude representations of rational numbers. This finding is further evidence that understanding conceptual change with magnitude representations of rational numbers is a necessary step prior to conceptual change with density concepts.

The results of the present study suggest that the use of latent variable mixture models are useful for the investigation of conceptual change with rational numbers, providing support for existing theories and models of the developmental patterns related to these difficult concepts (Merenluoto & Lehtinen, 2004; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Vosniadou & Verschaffel, 2004). More evidence is needed in order to confirm the appropriateness of the models identified in the present sample. In particular, the present study was only able to follow participants over a one-year period. The small amount of movement between classes in the LTA model suggests that a longer time frame is needed in order to capture more instances of conceptual change with rational numbers. As well, further studies of conceptual change with rational numbers, including those using latent variable mixture models, may benefit from the inclusion of measures of individual differences in the quality of learning experiences (Hannula & Lehtinen, 2005; McMullen, Hannula-Sormunen, & Lehtinen, 2013). While the participants' fairly young age was beneficial to view how conceptual knowledge of rational numbers develops at the start of formal teaching of rational numbers, the inclusion of older students in a similarly designed study would possibly provide more fruitful evidence of the process of conceptual change with these topics. Finally, the present study did not include Latent Class Analysis based on categorical variables, which has been used in previous studies of conceptual change using latent variable mixture modeling. Such an approach would allow for a more in depth look at specific conceptual models of rational numbers. More diverse items on a test of rational number conceptual knowledge would more readably support such approach, to the benefit of both researchers and educators.

Students will most likely always struggle with learning about the concepts of rational numbers that violate their everyday experiences with natural numbers. However, the present study does indicate that there are some students who are successful in transitioning to more mathematical appropriate concepts of rational numbers, in particular with concepts of the representations of the magnitudes of rational numbers.

Knowing this is possible should give hope to educators that progress can be made even in the most difficult of topics. This kind of more detailed knowledge about the developmental trajectories and patterns of sub-concepts of rational numbers can inform mathematics educators to develop more elaborated pedagogical practices which facilitate student conceptual change and help them to gradually acquire mathematically correct understanding of rational numbers. (see Confrey et al. 2009).

As researchers, we are encouraged to use all the tools available to us to understand as best we can the learning and developmental processes that students go through. The results of the present study suggest that latent variable mixture models should be considered a viable option for this.

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Table 2. Fit measures of 2-, 3-, and 4-class latent profile models at Times 1, 2, and 3

Number of Classes	Time 1 (N=251)				Time 2 (N=239)				Time 3 (N=221)			
	AIC	BIC	Entropy	BLRT (p)	AIC	BIC	Entropy	BLRT (p)	AIC	BIC	Entropy	BLRT (p)
2	1655.68	1701.51	0.98	<.001	1728.20	1774.03	0.98	<.001	1471.89	1516.06	0.99	<.001
3	1557.17	1617.10	0.93	<.001	1644.57	1704.50	0.92	<.001	1385.51	1443.28	0.93	<.001
4	1491.42	1565.46	0.95	<.001	1604.50	1678.54	0.92	<.001	1330.78	1402.14	0.96	<.001
5	1460.08	1548.33	0.94	<.001*	Estimation unstable (No convergence)				1302.40	1387.36	0.95	<.001*

*More random starts were needed to get a stable solution.