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Note

On the density of $(1, \leq \ell)$-locating-dominating codes in the infinite square grid

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ABSTRACT

Given a simple graph G, let $B_r(v)$ be the set of vertices at a distance of at most r from v. A vertex subset C is an $(r, \leq \ell)$-locating-dominating code of type A or $(r, \leq \ell)$-LDA code if the sets $I_r(F) = B_r(F) \cap C$ are distinct for all vertex subsets $F \subseteq V(G)$ of size at most ℓ that share the same vertices in C. Similarly, C is referred to as an $(r, \leq \ell)$-locating-dominating code of type B or $(r, \leq \ell)$-LDB code if the identifying sets $I_r(F)$ are distinct for all vertex subsets $F \subseteq V(G) \setminus C$ of size at most ℓ. In this article, we present optimal (with respect to density) $(1, \leq \ell)$-LDA and $(1, \leq \ell + 1)$-LDB codes in the infinite square grid for all $\ell \geq 2$. For $(1, \leq 2)$-LDB codes in infinite square grid, we show that the optimal code will have density between $[\frac{2}{5}, \frac{5}{11}]$.

1. Introduction and the main results

In the 1980s Slater [31,32] introduced *locating-dominating codes* as a variation and extension of the classical domination problem in graphs. Locating-dominating codes are motivated by their applications in *safeguard analysis of a facility* [31] or as *fault diagnosis in multiprocessor systems* [19]. In a multi-processor (vertices) system, some processors (codewords) are chosen to test whether the processor itself or if any processors in its r-neighborhood are faulty. It reports the symbol $0, 1$ or 2 depending on whether nothing is detected by the processor, the processor itself is faulty, or some of its adjacent processors are faulty, respectively. In this work, our motivation stems from a setup where we may observe multiple, up to $\ell \geq 1$, faulty processors.

Preliminaries: We follow West [36] for standard graph theoretic notation and terminology. The graphs we consider do not have parallel edges or loops, however they may be infinite.

A generalization of locating-dominating codes was introduced by Honkala et al. [15]. Given a graph G and a vertex v, the ball with center v and radius r is given by $B_r(v) = \{u \mid d(u, v) \leq r\}$, where $d(u, v)$ denotes the distance between u and v in graph G. Furthermore, for a vertex subset $F \subseteq V(G)$, the ball around F and radius r is given by

$$B_r(F) = \bigcup_{u \in F} B_r(u).$$

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A vertex subset $C \subseteq V(G)$ is called a *code* and its vertices are called *codewords* while the vertices of $V(G) \setminus C$ are called *non-codewords*. For each vertex $u \in V(G)$, *I-set* of a vertex is defined as

$$I_r(u) = B_r(u) \cap C.$$

Furthermore, an *I-set* of a vertex subset $F \subseteq V(G)$ is defined as

$$I_r(F) = B_r(F) \cap C = \bigcup_{u \in F} I_r(u).$$

In particular, C is called an *$(r, \leq \ell)$-locating-dominating code of type A*, or simply *$(r, \leq \ell)$-LDA code* if for any two distinct vertex subsets $X, Y \subseteq V(G)$ with cardinality at most ℓ, we have $X \cap C = Y \cap C \implies I_r(X) \neq I_r(Y)$. Similarly, C is called an *$(r, \leq \ell)$-locating-dominating code of type B*, or simply *$(r, \leq \ell)$-LDB code* if the I-sets $I_r(X)$ and $I_r(Y)$ are distinct for all distinct subsets $X, Y \subseteq V(G) \setminus C$ with cardinality at most ℓ. Note that by considering $X = \emptyset$, we may observe that both $(r, \leq \ell)$-LDA and -LDB codes are r-dominating sets (note that, $C \subseteq V(G)$ is an *r-dominating set* of G if for every vertex $u \in V(G)$, $B_r(u) \cap C \neq \emptyset$).

Related concepts: By definition, an $(r, \leq \ell)$-LDA code is, in particular, an $(r, \leq \ell)$-LDB code. When $r = \ell = 1$, the $(r, \leq \ell)$-LDA code and the $(r, \leq \ell)$-LDB code coincide with *locating-dominating code* (LD code for short), [31]. LD codes (also known as locating-dominating sets) are extensively studied from structural and algorithmic points of view (see [16] for an comprehensive online bibliography with more than 500 related papers). We note that each $(r, \leq \ell + 1)$-LDA or -LDB code is also an $(r, \leq \ell)$-LDA or -LDB code, respectively. Hence, for example, each $(1, \leq 2)$-LDA or -LDB code is also an LD code. Furthermore, in [18] the authors have introduced *self-* and *solid-locating-dominating codes*. They have shown that self-locating-dominating codes are between LD and $(1, \leq 2)$-LDA codes while solid-locating-dominating codes are between LD and $(1, \leq 2)$-LDB codes. Another related concept are *identifying codes* [19] and *$(r, \leq \ell)$-identifying codes* [20]. In particular, any identifying code is also an LD code. Furthermore, any $(r, \leq \ell)$-identifying code is also an $(r, \leq \ell)$-LDA code (and hence, also an $(r, \leq \ell)$-LDB code).

Several research articles [15,6,9,22,29,34,35,5] have studied $(r, \leq \ell)$-LDA and $(r, \leq \ell)$-LDB codes from structural and algorithmic perspectives. In particular, for various types of grids, $(r, \leq \ell)$-LDA and $(r, \leq \ell)$-LDB codes have been studied in [24,23,25,26] with a particular focus on the cases $r = \ell = 1$ in [33,7,13]. We note that previously many related concepts have been widely studied in infinite grids, see for example [1,8,11,27,17,10]. For improved understanding on the landscape of different types of locating-dominating codes, consult for example [21, Figure 6].

From the multi-processor perspective, the improvement from $(r, \leq \ell)$-LDA/LDB codes over the usual LD codes (or $(r, \leq 1)$-LD codes) is the capability to locate up to ℓ anomalies simultaneously. In particular, if we have a usual LD code and there are two anomalies, the system may indicate that there is only one anomaly and even in an incorrect location. On the other hand, $(r, \leq \ell)$-LDA/LDB codes both correctly indicate the number of anomalies (up to ℓ) and locate each of them. The difference between LDA and LDB is that in the case of LDA we assume that each vertex v may report only one of the values 0, 1 or 2 (where 0 means no anomalies in $B_r(v)$, 1 indicates an anomaly in $B_r(v) \setminus \{v\}$ but no anomalies in v and 2 is an anomaly in v). Furthermore, LDB codes can present different scenarios: For example, there can be no anomalies in the code vertices or we have a two-level system where we first take care of each anomaly at a code vertex and then solve the problems at non-code vertices [15].

Definitions: In this article, we focus on studying codes on the *infinite square grid* $\mathbf{G_4}$ whose vertices are of the form $v_{i,j}$ where $(i, j) \in \mathbb{Z}^2$ and two vertices $v_{i,j}$ and $v_{i',j'}$ are adjacent if the Euclidean distance between (i, j) and (i', j') is equal to 1.

Given a connected locally finite graph G and a vertex subset $S \subseteq V(G)$, we use a recent definition from [30] for the *density* of a code (vertex subset) C as follows:

$$D(C) = \inf \left\{ \limsup_{r \to \infty} \frac{|C \cap B_r(v)|}{|B_r(v)|} \text{ such that } v \in V(G) \right\}.$$

One may observe that the definition of the density of a code from [30] differs from the traditional definition for the same used in, for example, [33]. However, the authors of [30] have shown that their definition is, perhaps, better than the traditional one in some exceptional cases. Although in the case of square grids, the traditional notion of density of a code is equivalent with one we are using. A connected graph is said to have a *slow-growth* property, if there exists $s \in V(G)$ such that $\lim_{r \to \infty} \frac{|B_{r+1}(s)|}{|B_r(s)|} = 1$. In particular, the infinite square grid has the slow-growth property (as do many other commonly studied grids). Hence, by [30], we may use a relaxed equivalent definition for the density of $C \subseteq V(\mathbf{G_4})$:

$$D(C) = \limsup_{r \to \infty} \frac{|C \cap B_r(v_{0,0})|}{|B_r(v_{0,0})|}.$$

We further note, that in [30], the authors have shown that a graph having slow-growth property guarantees that discharging methods, where charge moves within a finite distance, work in the considered graph. We use discharging methods in our proofs.

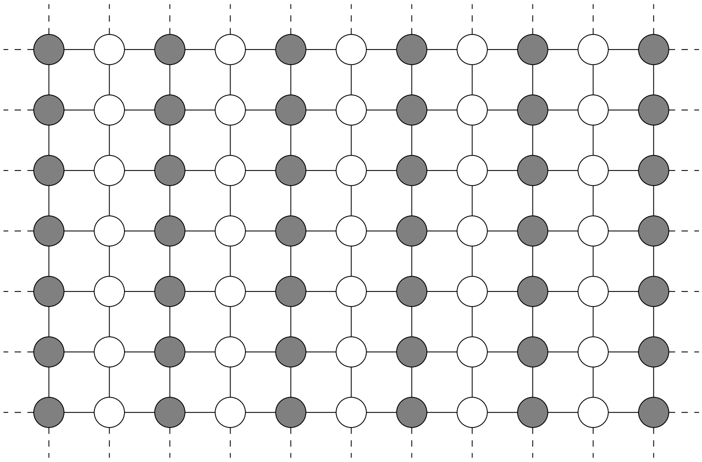

Fig. 1. The gray vertices form a $(1, \leq 2)$-LDA code with density $\frac{1}{2}$.

We will use the following lemma to prove some lower bound results. We note that originally the result has been proven for countably infinite graphs. Although, it has been later pointed out in [30] that a slightly stronger requirement of slow-growth property is necessary. However, this does not affect our use of lemma.

Lemma 1.1 *(Slater 2002, [33]). Given an infinite k-regular graph with the slow-growth property, we have $D(C) \geq \frac{2}{k+3}$ for any locating-dominating code C.*

Our contributions: In this article, we focus on studying the infinite square grid $\mathbf{G_4}$. An $(r, \leq \ell)$-LDA code (resp., $(r, \leq \ell)$-LDB code) is *optimal* if there does not exist any $(r, \leq \ell)$-LDA code (resp., $(r, \leq \ell)$-LDB code) with a strictly smaller density. Moreover, the density of an optimal $(r, \leq \ell)$-LDA code (resp., $(r, \leq \ell)$-LDB code) for the infinite square grid $\mathbf{G_4}$ is denoted by $D^A_{r,\ell}$ (resp., $D^B_{r,\ell}$). In 2002, Slater [33] computed the exact values of $D^A_{1,1}$ and $D^B_{1,1}$.

Theorem 1.2 *(Slater 2002, [33]). The density of an optimal $(1, \leq 1)$-LDA and -LDB code for the infinite square grid $\mathbf{G_4}$ is $\frac{3}{10}$.*

Our primary objective is to compute the densities $D^A_{1,\ell}$ and $D^B_{1,\ell}$ for all $\ell \geq 2$. In the case of $\ell \geq 3$, we extend our results on $D^A_{1,\ell}$ to $D^A_{r,\ell}$ for every $r \geq 1$. We summarize our main findings in the following two theorems.

Theorem 1.3. *The density of an optimal $(r, \leq \ell)$-LDA code for the infinite square grid $\mathbf{G_4}$ for all $\ell \geq 2$ is given by:*

(i) $D^A_{1,2} = \frac{1}{2}$,
(ii) $D^A_{r,\ell} = 1$, *for all $\ell \geq 3$ and $r \geq 1$.*

Theorem 1.4. *The density of an optimal $(r, \leq \ell)$-LDB code for the infinite square grid $\mathbf{G_4}$ for all $\ell \geq 2$ is given by:*

(i) $\frac{2}{5} \leq D^B_{1,2} \leq \frac{5}{11}$,
(ii) $D^B_{1,\ell} = \frac{1}{2}$, *for all $\ell \geq 3$,*
(iii) $D^B_{r,\ell} \leq \frac{r}{r+1}$, *for all $r, \ell \geq 1$.*

In the case of $r > 1$, even the density of $D^A_{r,1} = D^B_{r,1}$ is unknown. However, Lemma 1.1 implies a lower bound together with an understanding that a vertex is an $(r, \leq \ell)$-LDA or -LDB code in graph G if and only if it is a $(1, \leq \ell)$-LDA or -LDB code in the power graph G^r. As the power graph of square grid is a $(2r^2 + 2r)$-regular graph, Lemma 1.1 gives a lower bound of $\frac{2}{2r^2+2r+3}$ for $D^B_{r,\ell}$. Besides this lower bound, the only previously known results are from $(r, \leq 1)$-identifying codes which give upper bounds for $D^A_{r,1}$ (and hence also for $D^B_{r,1}$). We have included these results to Table 2 together with Theorems 1.3(ii) and 1.4(iii). We have included this table mainly to collect existing results and give anyone interested in studying this problem further a starting point.

2. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3(i). A $(1, \leq 2)$-LDA code of $\mathbf{G_4}$ with density $\frac{1}{2}$ is given in Fig. 1.

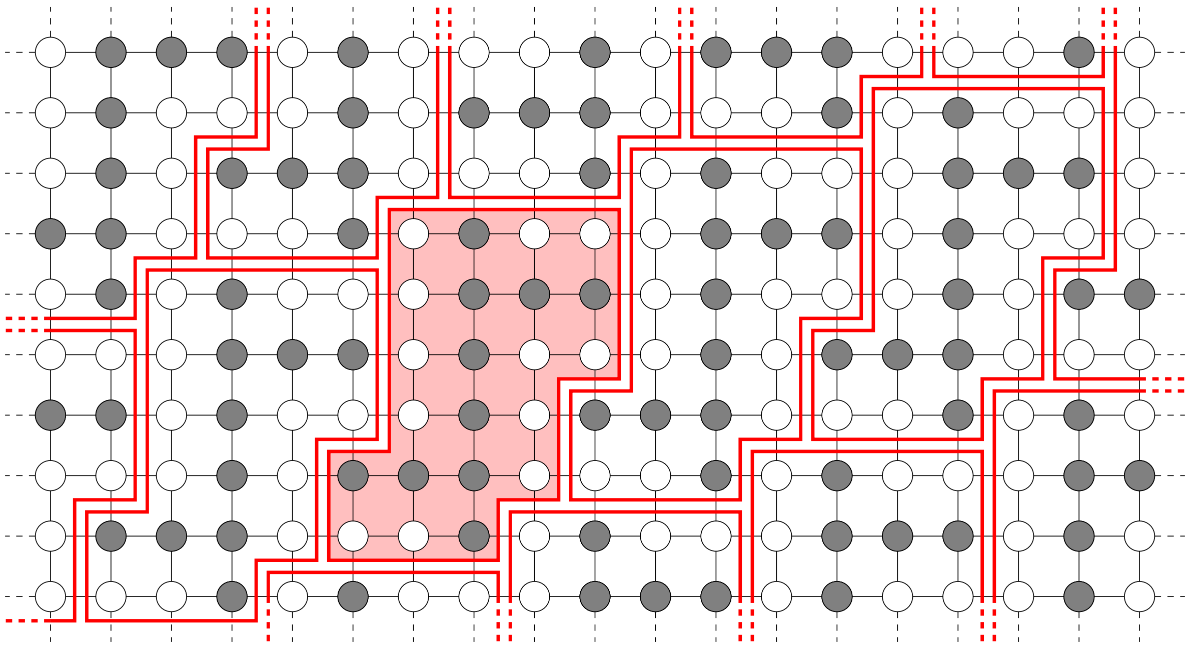

Fig. 2. The gray vertices form a $(1, \leq 2)$-LDB code with density $\frac{5}{11}$.

For the lower bound, suppose the contrary. That is, suppose that C is a $(1, \leq 2)$-LDA code of the infinite square grid $\mathbf{G}_4$ having density strictly less than $\frac{1}{2}$. Observe that, if each induced 4-cycle of the form $v_{i,j}v_{i+1,j}v_{i+1,j+1}v_{i,j+1}v_{i,j}$ contains at least 2 vertices of C, then the density of C is $\frac{1}{2}$ or more. That means, there must exist at least one induced 4-cycle of $\mathbf{G}_4$ that contains 1 or less vertices of C.

Without loss of generality, assume that the 4-cycle $v_{0,0}v_{1,0}v_{1,1}v_{0,1}v_{0,0}$ contains at most one vertex from C. In particular, assume that $v_{1,0}, v_{1,1}, v_{0,1} \notin C$. In that case, notice that $I_1(\{v_{1,1}, v_{2,2}\}) = I_1(v_{2,2})$, a contradiction. $\square$

Proof of Theorem 1.3(ii). It is enough to show that every vertex of the infinite square grid $\mathbf{G}_4$ must be part of any $(r, \leq 3)$-LDA code for any $r \geq 1$. Suppose the contrary, and thus, without loss of generality, let $v_{0,0}$ be a non-codeword. In this case, note that $I_r(\{v_{-1,-1}, v_{1,1}\}) = I_r(\{v_{-1,-1}, v_{0,0}, v_{1,1}\})$, a contradiction. $\square$

In the following, we describe discharging method which we use to prove the lower bound of Theorem 1.4(i). An interested reader may find more information about discharging method from [3], and in the specific context of infinite graphs and density from [30]. Let C be a $(1, \leq 2)$-LDB code of the infinite square grid $\mathbf{G}_4$. We start by assigning to each vertex an initial charge of

$$ch(x) = \begin{cases} \frac{3}{2}, & \text{if } x \in C \\ 0, & \text{if } x \in V(\mathbf{G}_4) \setminus C. \end{cases}$$

Let us next describe the discharging steps. We are going to describe two discharging rules R1 and R2, the first of which is as follows.

(R1) Each codeword $c \in C$ such that $|I_1(c)| < 5$, donates a charge of $\frac{3}{2(5-|I_1(c)|)}$ to each of its non-codeword neighbors.

A non-codeword u is identified as *charge-rich* if it has precisely one codeword neighbor.

(R2) Each charge-rich non-codeword u donates a charge of $\frac{1}{8}$ to each of its non-codeword neighbors.

Let the updated charge of a vertex $x \in V(\mathbf{G}_4)$ after performing (R1) and (R2) be denoted by $ch^*(x)$. We want to show that $ch^*(x) \geq 0$ for all $x \in C$ and $ch^*(x) \geq 1$ for all $x \in V(\mathbf{G}_4) \setminus C$. By the following lemma that will amount to proving that C has a density of at least $\frac{2}{5}$.

Lemma 2.1. *Let C be a $(1, \leq 2)$-LDB code. Let every vertex $x \in C$ have an initial charge $ch(x) = \alpha$ for some $\alpha > 0$ and every vertex $y \in V(\mathbf{G}_4) \setminus C$ have an initial charge $ch(y) = 0$. If we move charge between vertices so that charge is moved to at most distance 2 from its original location, and the updated charges are $ch^*(x) \geq 0$ and $ch^*(y) \geq 1$ for all $x \in C$ and for all $y \in V(\mathbf{G}_4) \setminus C$, then $D(C) \geq \frac{1}{1+\alpha}$.*

Proof. Observe that we move charge to at most distance 2 from a codeword. Thus, the initial charge within an n-radius ball is at least as large as the final charge within an $(n-2)$-radius ball with the same center point. Hence, when $ch^*(x) \geq 0$ for all $x \in C$ and $ch^*(y) \geq 1$ for a $(1, \leq 2)$-LDB code C, we obtain $\alpha|C \cap B_n(v_{0,0})| \geq |B_{n-2}(v_{0,0})| - |C \cap B_{n-2}(v_{0,0})| \geq |B_{n-2}(v_{0,0})| - |C \cap B_n(v_{0,0})|$. This implies $|C \cap B_n(v_{0,0})| \geq \frac{1}{1+\alpha}|B_{n-2}(v_{0,0})|$. Therefore, we have

$$\limsup_{n\to\infty} \frac{|C \cap B_n(v_{0,0})|}{|B_n(v_{0,0})|} \geq \limsup_{n\to\infty} \frac{\frac{1}{1+\alpha}|B_{n-2}(v_{0,0})|}{|B_n(v_{0,0})|} = \frac{1}{1+\alpha} \limsup_{n\to\infty} \frac{|B_{n-2}(v_{0,0})|}{|B_n(v_{0,0})|} = \frac{1}{1+\alpha}. \quad \square$$

We are now ready to give the proof of Theorem 1.4(i).

Proof of Theorem 1.4(i). A $(1, \leq 2)$-LDB code of $\mathbf{G}_4$ with density $\frac{5}{11}$ is given in Fig. 2 which proves the upper bound.

Let us next show the lower bound using the above-described discharging method. Our goal is to show that we have $ch^*(x) \geq 0$ and $ch^*(y) \geq 1$ for any $(1, \leq 2)$-LDB code C, all vertices $x \in C$ and $y \in V(\mathbf{G}_4) \setminus C$. After this, the claim follows from Lemma 2.1 together with choice $\alpha = \frac{3}{2}$. First of all, note that if a codeword c has $k \geq 1$ non-codewords in its neighborhood, then c donates in (R1) a total charge of

$$k \times \frac{3}{2(5 - |I_1(c)|)} = k \times \frac{3}{2(5-(5-k))} = \frac{3}{2}.$$

This implies that after donating charge, the updated charge of each codeword c is

$$ch^*(c) \geq ch(c) - \frac{3}{2} = 0.$$

Next let u be a non-codeword with exactly one codeword neighbor c. Observe that each of the three neighbors of c, other than u, must all be codewords. Otherwise, for any non-codeword neighbor $z \neq u$ of c, $I_1(z) = I_1(\{u, z\})$, which violates the definition of $(1, \leq 2)$-LDB codes. Hence, u receives a charge of $\frac{3}{2}$ after the application of rule (R1) and donates a total charge of $3 \times \frac{1}{8}$ due to (R2). Thus, $ch^*(u) = \frac{3}{2} - \frac{3}{8} = \frac{9}{8} \geq 1$.

Hence, we are left to show that $ch^*(u) \geq 1$ when u is a non-codeword having either $2, 3$, or 4 codeword neighbors. For convenience of calculation, let us assume, without loss of generality, that $u = v_{0,0}$ is the non-codeword.

i) Suppose $u = v_{0,0}$ has exactly two codeword neighbors. Thus, without loss of generality, we may consider two cases where either vertices $v_{0,1}$ and $v_{1,0}$, or vertices $v_{0,1}$ and $v_{0,-1}$ are codewords.
   - Let $v_{0,1}$ and $v_{1,0}$ be the codeword neighbors of $v_{0,0}$. Observe that $v_{1,1}$ must be a codeword, otherwise $I_1(v_{1,1}) = I_1(\{v_{0,0}, v_{1,1}\})$ which contradicts the definition of $(1, \leq 2)$-LDB codes. Hence, $|I_1(v_{0,1})|, |I_1(v_{1,0})| \geq 2$, and thus, both $v_{0,1}$ and $v_{1,0}$ donate at least a charge of $\frac{1}{2}$ to $v_{0,0}$ due to the application of (R1). Thus, $ch^*(u) = ch^*(v_{0,0}) \geq 1$.
   - Let $v_{0,1}$ and $v_{0,-1}$ be the codeword neighbors of $v_{0,0}$. Note that both $v_{0,1}$ and $v_{0,-1}$ will donate at least $\frac{3}{8}$ charge to $v_{0,0}$. Hence, we may assume, without loss of generality, that $v_{0,0}$ obtains at least as much charge from $v_{0,-1}$ as it obtains from $v_{0,1}$, that is, we are assuming that $|I_1(v_{0,-1})| \geq |I_1(v_{0,1})|$.
     Notice that, if $v_{0,-1}$ donates a charge of at least $\frac{3}{4}$, or if both $v_{0,1}$ and $v_{0,-1}$ donate a charge of $\frac{1}{2}$ to $v_{0,0}$, then $ch^*(u) = ch^*(v_{0,0}) \geq 1$.
     Thus, we need to handle the case when $v_{0,1}$ donates a charge of $\frac{3}{8}$ and $v_{0,-1}$ donates a charge of $\frac{1}{2}$ or $\frac{3}{8}$ to $v_{0,0}$. In such a scenario, we know that all neighbors of $v_{0,1}$ are non-codewords, and all but possibly one neighbor of $v_{0,-1}$ are non-codewords. Therefore, $v_{-1,1}$ is a non-codeword and we may assume, without loss of generality, that also $v_{-1,-1}$ is a non-codeword. Thus, $v_{-2,0}$ is a codeword, as otherwise the non-codeword neighbor $v_{-1,0}$ of $v_{0,0}$ will not be dominated. In particular, $v_{-1,0}$ is a charge-rich neighbor of $v_{0,0}$ and donates charge of $\frac{1}{8}$ to $v_{0,0}$ due to the application of rule (R2).
     Similarly, if $v_{1,-1}$ is a non-codeword, then it will force $v_{1,0}$ to be charge-rich and donate $\frac{1}{8}$ to $v_{0,0}$ due to the application of rule (R2).
     Hence, in one case $v_{0,0}$ receives charges of at least $\frac{3}{8}$ from both $v_{0,1}$ and $v_{0,-1}$, and $\frac{1}{8}$ from both $v_{-1,0}$ and $v_{1,0}$. In the other case, $v_{0,0}$ receives charges of at least $\frac{3}{8}$ from $v_{0,1}$, $\frac{1}{2}$ from $v_{0,-1}$, and $\frac{1}{8}$ from $v_{-1,0}$. In either case we have $ch^*(u) = ch^*(v_{0,0}) \geq 1$.

ii) Suppose next that $u = v_{0,0}$ has 3 or more codeword neighbors. Since $v_{0,0}$ receives at least $\frac{3}{8}$ charge from each of its codeword neighbors, we have $ch^*(u) = ch^*(v_{0,0}) \geq 3 \times \frac{3}{8} = \frac{9}{8} \geq 1$.

This completes the proof. $\square$

Proof of Theorem 1.4(ii). A $(1, \leq 3)$-LDB code of $\mathbf{G_4}$, for all $\ell \geq 3$, with density $\frac{1}{2}$ is given in Fig. 3 as an upper bound. Indeed, with $r = 1$ we obtain $\frac{r}{r+1} = \frac{1}{2}$.

For the lower bound, let C be a $(1, \leq 3)$-LDB code, for all $\ell \geq 3$, of the infinite square grid $\mathbf{G}_4$. We again apply a discharging method to prove our claimed lower bound. For this, we will give each codeword an initial charge of 1 and every

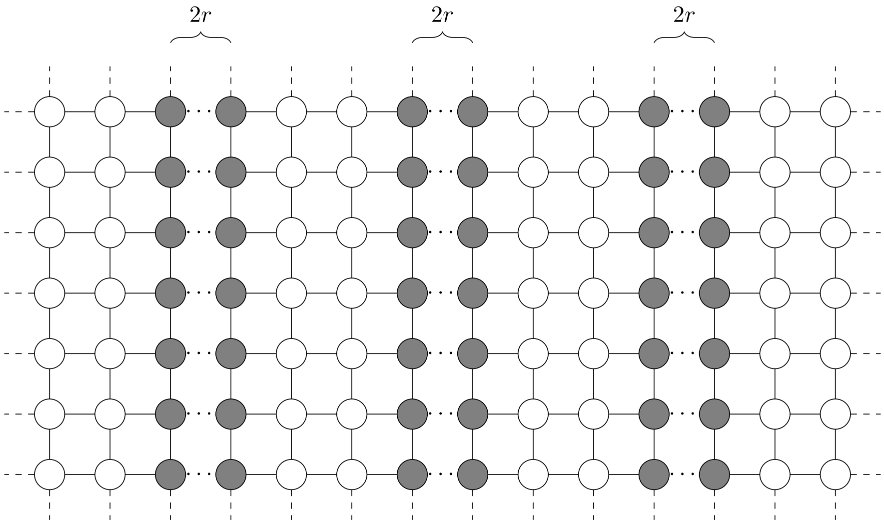

Fig. 3. The gray vertices form an $(r, \leq \ell)$-LDB code with density $\frac{r}{r+1}$ for any $r, \ell \geq 1$.

other vertex an initial charge of 0. Now, the claimed lower bound follows from Lemma 2.1 together with choice $\alpha = 1$ if we can find a charge shifting scheme which leads to a final charge of at least 0 for each codeword and at least 1 for each non-codeword. Next let us assign an initial charge of 1 to all the codewords $c \in C$. We will do this by having each codeword $c \in C$ such that $|I_1(c)| < 5$, donate a charge of $\frac{1}{5-|I_1(c)|}$ to each of its non-codeword neighbors. We notice that we shift charge to a distance of at most 1 from its original location and hence, we may apply Lemma 2.1. If a codeword c has $k \geq 1$ non-codewords in its neighborhood, then c donates a total charge of

$$k \times \frac{1}{5 - |I_1(c)|} = k \times \frac{1}{5 - (5 - k)} = 1.$$

This implies that after donation, the updated charge of each codeword is at least 0.

Let u be a non-codeword vertex in $V(\mathbf{G}_4)$. If u has exactly one codeword neighbor c, then all other neighbors of c must be codewords. To see why, assume there is a non-codeword vertex v other than u in the neighborhood of c. In this case, we observe that $I_1(v) = I_1(\{u, v\})$, which contradicts the definition of $(1, \leq 3)$-LDB codes, and hence that of $(1, \leq \ell)$-LDB codes for all $\ell \geq 3$. Therefore, u receives a charge of 1 from c.

If u has exactly two codeword neighbors c_1, c_2, then at least one of these codeword neighbors must have all of its neighbors other than u as codewords. If not, assume there is a non-codeword neighbor $v \neq u$ of c_1 and a non-codeword neighbor $w \neq u$ of c_2. Here we can see that $I_1(\{v, w\}) = I_1(\{u, v, w\})$, which is a contradiction to the definition of $(1, \leq 3)$-LDB codes and hence that of $(1, \leq \ell)$ for all $\ell \geq 3$. Thus, u receives a charge of at least $\frac{5}{4} \geq 1$.

Let u have exactly three codeword neighbors. Assume, without loss of generality, that $u = v_{0,0}$ and its codeword neighbors are $v_{0,-1}, v_{1,0}, v_{0,1}$. Suppose that both $v_{1,1}$ and $v_{1,-1}$ are non-codewords. Thus, $I_1(\{v_{1,1}, v_{1,-1}\}) = I_1(\{u, v_{1,1}, v_{1,-1}\})$, and hence one of the vertices between $v_{1,1}$ and $v_{1,-1}$ must be a codeword. Without loss of generality, let us assume that $v_{1,1}$ is a codeword. Next, suppose that both $v_{1,-1}$ and $v_{-1,1}$ are non-codewords. Thus, $I_1(\{v_{1,-1}, v_{-1,1}\}) = I_1(\{u, v_{1,-1}, v_{-1,1}\})$, and thus one of the vertices $v_{1,-1}, v_{-1,1}$ is a codeword. If $v_{1,-1}$ (resp., $v_{-1,1}$) is a codeword, then u receives at least $\frac{1}{2}$ from $v_{1,0}$ (resp., $v_{0,1}$). As u receives at least $\frac{1}{4}$ from its other codeword neighbors, u receives a total charge of at least 1 from its codeword neighbors.

If u has 4 codeword neighbors, then it receives a total charge of at least $4 \times \frac{1}{4} = 1$. □

Proof of Theorem 1.4(iii). An $(r, \leq \ell)$-LDB code C of $\mathbf{G_4}$, for all $r, \ell \geq 1$, with density $\frac{r}{r+1}$ is given in Fig. 3 as an upper bound. Indeed, the density is $\frac{r}{r+1}$ as the figure has the pattern of $2r$ gray vertices followed by two white vertices in each row. Moreover, it is an $(r, \leq \ell)$-LDB code since each white vertex v is within distance r to such a gray vertex w that $w \in I_r(F)$ for $F \subseteq V(\mathbf{G_4}) \setminus C$ if and only if $v \in F$. □

3. Conclusions

In this article, we have given exact density for optimal $(1, \leq \ell)$-LDA codes for each $\ell \geq 2$ and for optimal $(1, \leq \ell)$-LDB codes for each $\ell \geq 3$ in the infinite square grid. For $(1, \leq 2)$-LDB we have given lower and upper bounds. We note that our proof of lower bound in Theorem 1.3(i) giving the exact lower bound $\frac{1}{2}$ for the density of $(1, \leq 2)$-LDA codes in the infinite square grid is actually a generalization of [10, Theorem 1] which gave the same lower bound of $\frac{1}{2}$ for the density

Table 1
Upper and lower bounds for the densities of optimal $(1, \le \ell)$-LDA and $(1, \le \ell)$-LDB codes in four types of infinite grids.

ℓ	Hexagonal grid ($\mathbf{G}_3$)		Square grid ($\mathbf{G}_4$)		Triangular grid ($\mathbf{G}_6$)		King grid ($\mathbf{G}_8$)	
	$D^A_{1,\ell}(\mathbf{G}_3)$	$D^B_{1,\ell}(\mathbf{G}_3)$	$D^A_{1,\ell}(\mathbf{G}_4)$	$D^B_{1,\ell}(\mathbf{G}_4)$	$D^A_{1,\ell}(\mathbf{G}_6)$	$D^B_{1,\ell}(\mathbf{G}_6)$	$D^A_{1,\ell}(\mathbf{G}_8)$	$D^B_{1,\ell}(\mathbf{G}_8)$
$=1$	$\frac{1}{3}$ [13]	$\frac{1}{3}$ [13]	$\frac{3}{10}$ [33]	$\frac{3}{10}$ [33]	$\frac{17}{57}$ [7]	$\frac{17}{57}$ [7]	$\frac{1}{5}$ [13]	$\frac{1}{5}$ [13]
$=2$	$[\frac{1}{3}, \frac{2}{3}]$ [13], [11]	$[\frac{1}{3}, \frac{2}{3}]$ [13], [11]	$\frac{1}{2}$ (see Theorem 1.3(i))	$[\frac{2}{5}, \frac{5}{11}]$ (see Theorem 1.4(i))	$\frac{1}{3}$ [4]	$\frac{1}{3}$ [4]	$\frac{1}{3}$ [25]	$\frac{1}{3}$ [25]
$=3$	$[\frac{1}{3}, 1]$ [13]	$[\frac{1}{3}, 1]$ [13]	1 (see Theorem 1.3(ii)	$\frac{1}{2}$ (see Theorem 1.4(ii))	1 [4]	$\frac{4}{7}$ [4]	1 [23]	$\frac{3}{5}$ [26]
≥ 4	1 [22]	$[\frac{1}{2}, 1]$ [22]	1 (see Theorem 1.3(ii))	$\frac{1}{2}$ (see Theorem 1.4(ii))	1 [4]	$\frac{2}{3}$ [4]	1 [23]	$\frac{2}{3}$ [23]

Table 2
Upper and lower bounds for the densities of optimal $(r, \le \ell)$-LDA and $(r, \le \ell)$-LDB codes in the infinite square grid for $r \ge 2$.

ℓ	Square grid ($\mathbf{G}_4$), $r=2$		Square grid ($\mathbf{G}_4$), $r=3$		Square grid ($\mathbf{G}_4$), $r=4$	
	$D^A_{r,\ell}(\mathbf{G}_4)$	$D^B_{r,\ell}(\mathbf{G}_4)$	$D^A_{r,\ell}(\mathbf{G}_4)$	$D^B_{r,\ell}(\mathbf{G}_4)$	$D^A_{r,\ell}(\mathbf{G}_4)$	$D^B_{r,\ell}(\mathbf{G}_4)$
$=1$	$[\frac{2}{15}, \frac{5}{29}]$ (see Lemma 1.1), [14]	$[\frac{2}{15}, \frac{5}{29}]$ (see Lemma 1.1), [14]	$[\frac{2}{27}, \frac{1}{8}]$ (see Lemma 1.1), [2]	$[\frac{2}{27}, \frac{1}{8}]$ (see Lemma 1.1), [2]	$[\frac{2}{43}, \frac{8}{85}]$ (see Lemma 1.1), [2]	$[\frac{2}{43}, \frac{8}{85}]$ (see Lemma 1.1), [2]
$=2$	$[\frac{2}{15}, 1]$ (see Lemma 1.1)	$[\frac{2}{15}, \frac{2}{3}]$ (Lemma 1.1, Theorem 1.4(iii))	$[\frac{2}{27}, 1]$ (see Lemma 1.1)	$[\frac{2}{27}, \frac{3}{4}]$ (Lemma 1.1, Theorem 1.4(iii))	$[\frac{2}{43}, 1]$ (see Lemma 1.1)	$[\frac{2}{43}, \frac{4}{5}]$ (Lemma 1.1, Theorem 1.4(iii))
≥ 3	1 (see Theorem 1.3(ii))	$[\frac{2}{15}, \frac{2}{3}]$ (Lemma 1.1, Theorem 1.4(iii))	1 (see Theorem 1.3(ii)	$[\frac{2}{27}, \frac{3}{4}]$ (Lemma 1.1, Theorem 1.4(iii))	1 (see Theorem 1.3(ii))	$[\frac{2}{43}, \frac{4}{5}]$ (Lemma 1.1, Theorem 1.4(iii))

of $(1, \le 2)$-identifying codes in the infinite square grid. As every $(1, \le 2)$-identifying code is also a $(1, \le 2)$-LDA code, our lower bound result yields the previous results on identifying codes as corollary. Additionally, some of our results generalize for larger r.

(1) A natural open question is to close the gap in the bounds reported in Theorem 1.4(i) of the density for $(1, \le 2)$-LDB codes of the infinite square grid which seems challenging to improve upon. Notice that the gap is $\frac{3}{55} \approx 0.054$. This gap shows that the upper bound proof guarantees a solution that achieves an approximation ratio of $\frac{25}{22} \approx 1.14$ relative to the optimum.

(2) In relation to the study of other infinite grids, $(1, \le \ell)$-locating dominating codes have been studied for the infinite 8-regular grid (also known as the king grid) in [24–28], for infinite triangular grid in [4] and in [7] for $\ell = 1$, and for infinite hexagonal grid for $\ell = 1$ in [13]. The corresponding results are summarized in Table 1. The cells of the table which lack tight bounds can be considered as open problems.

(3) We have included in Table 2 the known upper and lower bounds of $D^A_{r,\ell}$ and $D^B_{r,\ell}$ for $r \ge 2$. We note that not much is known in this case. In particular, studying the values of $D^A_{r,1} = D^B_{r,1}$ seems an interesting starting point. In [12], the authors have shown that there are infinitely many values of $r \ge 18$ for which there exists an $(r, \le 2)$-identifying code with density $\frac{r+1}{8r+4}$. For the values of r for which this construction exists, we obtain upper bounds for $(r, \le \ell)$-LDA/LDB codes. Besides the case with $\ell = 1$, for LDA-codes only the case with $\ell = 2$ is interesting due to Theorem 1.3(ii). In the case of LDB-codes, also the case of $(r, \le 2r^2 + 2r + 1)$-LDB codes seems interesting where $2r^2 + 2r$ is equal to the degree of every vertex in the rth power graph of the square grid. The interest for this special case rises from [22] where it has been noted that in these cases the definition of LDB-codes is equal with the definitions of some other concepts. It seems quite plausible that the construction of Theorem 1.4(iii) is tight for this case.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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