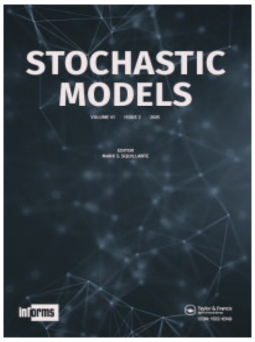

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One-to-one correspondences between discrete multivariate stationary, self-similar, and stationary increment fields

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ABSTRACT

In this article, we consider three important classes of n-variate fields indexed by the set of N dimensional integers, namely stationary, stationary increment, and self-similar fields. We connect these classes through bijective transformations. The one-to-one correspondence between stationary and self-similar fields, where the index of self-similarity is a tuple of positive definite matrices, is given by a version of the Lamperti transformation. In addition, we introduce generalized AR(1) type equations, whose unique stationary solutions are obtained via these transformations. Last, we apply the transformations in order to construct multivariate stationary fractional Ornstein-Uhlenbeck fields of the first and second kind, including a brief simulation study of bivariate Ornstein-Uhlenbeck sheets.

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1. Introduction

Stationary, self-similar, and stationary increment processes form undoubtedly some of the most central classes of stochastic processes. Our focus is on processes that are indexed by the set of N dimensional integers and taking values in the n dimensional real space. Henceforth, we call this type of objects multivariate fields, whereas by (univariate) stochastic process we refer to collections of random variables indexed by the one dimensional set $T \in \{\mathbb{R}, \mathbb{Z}\}$, although the wider meaning of the term allows vast variation of state and parameter spaces. The cases $T = \mathbb{R}$ and $T = \mathbb{Z}$ (or more generally $T = \mathbb{R}^N$ and $T = \mathbb{Z}^N$) are called continuous and discrete, respectively. For some of the numerous potential real-world applications of the random fields that are of interest in this article, see e.g., Biermé et al. [1, 18, 24], and references therein. In addition, we discuss the applied aspect of our research in Subsection 3.1.

The considered type of stationarity is the strict one, meaning that the multidimensional distributions are invariant under uniform translations in the parameter space. In order to discuss stationarity of increments in the context

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of fields, we first need a notion for multidimensional increments. We adapt the definition applied e.g. in Makogin and Mishura[18], where also different forms of stationarity of increments was investigated. Moreover, self-similarity refers to invariance of multidimensional distributions under appropriate scalings in the parameter and state spaces. The conventional notion of self-similarity of univariate processes have been extended to more generel settings in various ways. In the case of continuous time multivariate processes, different types of operator self-similarity have been considered e.g. in Hudson and Mason[9, 12, 23]. In Genton et al.[8] the authors formalized the concept of multi-self-similarity for anisotropic univariate fields with the parameter space $\mathbb{R}^N$. Another approach toward anisotropy called operator-scaling random fields have been investigated e.g. in Biermé et al.[1, 3, 16]. See also Kolodyński and Rosiński[11], where the notion of G-self-similarity allows a large spectrum of transformations of the parameter and state spaces. For an overview of univariate self-similar processes, we refer to Embrechts[6].

It was already shown by Lamperti [13] that there exists a one-to-one correspondence between stationary and self-similar processes, which nowadays is known as Lamperti transformation. For an overview of Lamperti transformation, its applications and some of its variations, see Flandrin et al.[7] In the case of continuous multi-self-similar univariate fields, a generalization of Lamperti transformation was introduced in Genton et al.[8], and in the case of discrete fields in Voutilainen et al.[26]. The two-parameter continuous case was discussed slightly earlier in Terdik and Woyczyński[24]. See also the recent paper by Davydov and Paulauskas[5], where Lamperti transformation was studied in a general setting related to the one in Kolodyński and Rosiński[11].

It is acknowledged that Langevin (CAR(1)) equation can be regarded as the continuous time analogue of discrete time AR(1) equation. In their classical forms, the noise can be derived from Brownian motion. One option to generalize these equations is to replace the Brownian driver with some other stationary increment noise. This approach was adapted e.g. in Voutilainen [25] and Voutilainen et al.[27], where generalized AR(1) and Langevin equations were studied for multivariate processes. The applied tools include a generalization of Lamperti transformation mapping from stationary to operator self-similar multivariate processes. Particularly, it was shown that essentially all stationary processes can be characterized as solutions to these equations. Similar results for univariate fields have been established in Voutilainen et al.[26].

We recall that fractional Brownian motion is self-similar and it has stationary increments being the unique centered Gaussian process possessing these properties. We immediately obtain two alternative ways to define a stationary process connected to fractional Brownian motion.

(i) as the unique stationary solution to AR(1) or Langevin equation driven by fBm.
(ii) via Lamperti transformation of fBm.

The resulting stationary processes are called fractional Ornstein-Uhlenbeck processes of the first and second kind, respectively, and they have received a lot of attention in the literature. For details, see e.g., Cheridito et al.[2]

The remaining content of the article is organized as follows. Subsection 2.1 introduces the applied notation. In Subsection 2.2, we provide notions for multivariate stationary, stationary increment, and self-similar fields, where the index Θ of self-similarity is a tuple of positive definite matrices. The given definition of self-similarity covers operator self-similarity of multivariate processes and multi-self-similarity of univariate fields as its special cases. A generalized Lamperti transformation $\mathcal{L}_\Theta$ gives a one-to-one correspondence between stationary fields and Θ-self-similar fields. We construct transformations $\mathcal{M}_\Theta$ and $\mathcal{M}_\Theta^{-1}$ mapping from Θ-self-similar fields to stationary increment fields, and vice versa. The mapping is bijective restricted on a subset of stationary increment fields that includes e.g. multivariate fractional Brownian sheets. As a corollary, the composition $\mathcal{M}_\Theta \circ \mathcal{L}_\Theta$ gives a one-to-one correspondence between stationary fields and the class of stationary increment fields.

Moreover, we introduce equations of AR(1) type, where the noise is a general stationary increment field. We show that stationary fields can be characterized as solutions to these equations, and the connection between the stationary solution and the noise is given by the composition of the two previous transformations.

In Subsection 2.3, we illustrate how transformations $\mathcal{L}_\Theta^{-1} \circ \mathcal{M}_\Theta^{-1}$ and $\mathcal{L}_\Theta^{-1}$ can be utilized in construction of stationary fractional Ornstein-Uhlenbeck fields of the first and second kind.

Subsection 3.1 is for discussion of possible fields of applications of the theoretical results. This is followed by a short numerical study of bivariate Ornstein-Uhlenbeck sheets in Subsection 3.2.

The proofs are postponed to Section 4.

2. Results

We begin with introducing the notation in Subsection 2.1. Subsection 2.2 consists of the essential definitions and the main results, which are applied in Subsection 2.3 in order to formally define multivariate stationary fractional Ornstein-Uhlenbeck fields of the first and second kind.

2.1. Notation

The notation $\mathbb{N}$ stands for strictly positive integers. Let X and Y be random vectors of length n. Then the equality of the distributions is denoted by $X \overset{\text{law}}{=}$

Y. Similarly, if $X = (X_t)_{t\in\mathbb{Z}^N}$ and $Y = (Y_t)_{t\in\mathbb{Z}^N}$ are n-dimensional fields with equal finite dimensional distributions, we write $X \stackrel{\text{law}}{=} Y$. That is, for any finite collection $t_1, \ldots, t_k \in \mathbb{Z}^N$ the joint distributions of k-tuples $\{X_{t_1}, \ldots X_{t_k}\}$ and $\{Y_{t_1}, \ldots Y_{t_k}\}$ of random vectors are equal. The l^2-vector norm and the corresponding induced matrix norm are denoted by $\|\cdot\|$. The set of symmetric $n \times n$-matrices is denoted by S^n. Moreover, the set of (symmetric) positive definite $n \times n$ matrices is denoted by S^n_+.

Occasionally, we also make use of the following notation. Let $u \subseteq \{1, \ldots, N\}$ and $t \in \mathbb{Z}^N$. The cardinality of u is denoted by $|u|$ and the complement of u with respect to $\{1, \ldots, N\}$ is denoted by $-u$. Moreover, the notation t_u stands for picking the elements of t, whose indices belong to u meaning that t_u is a $|u|$-vector. For example, if $u = \{1, 4, 5\}$ and $t \in \mathbb{Z}^5$, then $t_u = (t_1, t_4, t_5)$. In particularly in Lemma 2.34 and Definition 2.20, the notation is also applied in the context of multiple sums. If $j = (j_1, \ldots, j_N)$ is a multi-index, then for example, we write

$$\sum_{j_u=\mathbf{1}}^{t_u}$$

meaning that if $l \in u$, then j_l runs from one to t_l in the summation.

As a consequence of multidimensionality of the index and state spaces, we are working with parameters that are tuples of matrices.

Definition 2.1. *By $(S^n_+)^N$ we denote the set of N-tuples of positive definite $n \times n$-matrices. That is, $\Theta = (\Theta_1, \ldots, \Theta_N) \in (S^n_+)^N$ if $\Theta_j \in S^n_+$ for all $j \in \{1, \ldots, N\}$. We use similar notation also for N-tuples of different types of objects, e.g. n-vectors. Furthermore, $(S^n_+)^N_c$ denotes the subset of $(S^n_+)^N$ such that if $\Theta \in (S^n_+)^N_c$, then Θ has pairwisely commuting elements with respect to the standard matrix product.*

The sets $(S^n_+)^N$ and $(S^n_+)^N_c$ are fundamental for us. For this reason, the most of definitions and results are stated under the assumption $\Theta \in (S^n_+)^N$ (or $\Theta \in (S^n_+)^N_c$), although some of these could easily be formulated for larger classes of N-tuples of matrices. Also the following bivariate operators acting on tuples of integers, vectors and matrices play an essential role in our work.

Definition 2.2. *Let $t \in \mathbb{Z}^N$ and $\Theta \in (S^n)^N$. We define an operator $* : \mathbb{Z}^N \times (S^n)^N \to S^n$ by*

$$t * \Theta = \sum_{j=1}^{N} t_j \Theta_j.$$

In addition, if $X = (X_1, \ldots, X_N) \in (\mathbb{R}^n)^N$ is a N-tuple of n-vectors, we define an operator $\star : (S^n)^N \times (\mathbb{R}^n)^N \to \mathbb{R}^N$ by

$$\Theta \star X = \sum_{j=1}^{N} \Theta_j X_j.$$

Remark 2.3. When $n = 1$ the above definitions reduce to standard inner products of vectors: $t * \Theta = \langle t, \Theta \rangle$ and $\Theta \star X = \langle \Theta, X \rangle$.

2.2. Multivariate stationary, self-similar and stationary increments fields

We start by providing notions of stationarity and self-similarity in our setting. We consider multivariate fields that are stationary in the strict sense, and we refer to such fields simply as ”stationary fields” in the sequal. The usual definition of stationary processes extends directly to multivariate fields.

Definition 2.4. *A field $X = (X_t)_{t \in \mathbb{Z}^N} = (X^{(1)}_{t_1,\ldots,t_N}, \ldots, X^{(n)}_{t_1,\ldots,t_N})^T_{t \in \mathbb{Z}^N}$ is stationary if*

$$(X_{t+s})_{t \in \mathbb{Z}^N} \overset{law}{=} (X_t)_{t \in \mathbb{Z}^N}$$

for every $s \in \mathbb{Z}^N$.

The notion of self-similarity is a more delicate matter. First of all, we need to ensure that our discrete parameter space is closed under appropriate scalings. This is done by choosing ”exponential clocks” for self-similar fields. Second, we want to take account of fields that exhibit different types of self-similarity in different dimensions of the parameter and state spaces. Thus, the following definition combines the concepts of multi-self-similarity of univariate fields and operator self-similarity of multivariate processes.

Definition 2.5. *Let $\Theta \in (S^n_+)^N$. A field $Y = (Y_{e^t})_{t \in \mathbb{Z}^N} = (Y^{(1)}_{e^{t_1},\ldots,e^{t_N}}, \ldots, Y^{(n)}_{e^{t_1},\ldots,e^{t_N}})_{t \in \mathbb{Z}^N}$ is Θ-self-similar if*

$$(Y_{e^{t+s}})_{t \in \mathbb{Z}^N} \overset{law}{=} (e^{s * \Theta} Y_{e^t})_{t \in \mathbb{Z}^N}$$

for every $s \in \mathbb{Z}^N$.

Next, we discuss how Definition 2.5 is connected to definitions existing in the literature. Under Definition 2.5,

$$Y_{e^{t_1},\ldots,e^{t_j+s_j},\ldots,e^{t_N}} \overset{\text{law}}{=} e^{s_j \Theta_j} Y_{e^{t_1},\ldots,e^{t_N}}.$$

Let $Q_j \Lambda_j Q_j^T$ be an eigendecomposition of Θ_j with eigenvalues $\lambda_{j,k}$, then

$$e^{s_j \Theta_j} = Q_j \text{diag}\left(e^{s_j \lambda_{j,k}}\right) Q_j^T \to 0$$

as $s_j \to -\infty$. Consequently, under the usual assumption of stochastic continuity, a self-similar field $Y_{r_1,\dots,r_N}$ that is defined also on the hyperplanes $r_j = 0$ satisfies $Y_{r_1,\dots,r_N} = 0$ on these hyperplanes, as it is expected.

Let us consider Definition 2.5 with the index set $\mathbb{R}^N$. We note that if $\Theta \in (S_+^n)_c^N$, then change of variables $e^{t_j} = r_j$ and $e^{s_j} = a_j$ yield

$$\left(Y_{a_1 r_1,\dots,a_N r_N}\right)_{r\in(0,\infty)^N} \stackrel{\text{law}}{=} a_1^{\Theta_1} \dots a_N^{\Theta_N} (Y_r)_{r\in(0,\infty)^N}. \tag{1}$$

When $n = 1$, this corresponds to the definition of multi-self-similarity of univariate fields in Genton et al.[8], where self-similarity indices Θ_j are positive real numbers representing self-similarity in different dimensions of the parameter space.

For index set $\mathbb{R}$ ($N = 1$), Definition 2.5 gives

$$(Y_{ar})_{r\in(0,\infty)} \stackrel{\text{law}}{=} a^{\Theta} (Y_r)_{r\in(0,\infty)},$$

which corresponds to a type of operator self-similarity of vector-valued processes. This extends in a natural way the conventional notion of self-similarity of scalar random processes, where $n = N = 1$ and Θ is a positive real number. For notions of operator self-similarity, see Sato[23] and references therein. For an account on self-similarity in the basic setting, we refer to Embrechts[6].

A well-known example of a scalar-valued multi-self-similar field is the fractional Brownian sheet (fBs) $B^H = (B_t^H)_{t\in\mathbb{R}^N}$ with Hurst multi-index $H = (H_1,\dots,H_N)$, $0 < H \leq 1$, introduced in Kamont[10]. FBs is the centered Gaussian field with covariance

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2^N} \prod_{j=1}^{N} \left(|t_j|^{2H_j} + |s_j|^{2H_j} - |t_j - s_j|^{2H_j}\right),$$

and it is multi-self-similar with $\Theta = H$. Note that $B_t^H = 0$ (a.s) on the hyperplanes with $t_l = 0$ for some $l = 1,\dots,N$.

Let $\Theta_j = \operatorname{diag}(\Theta_j^{(k)})$ with positive entries. Then from (1), we get

$$\left(Y_{a_1 r_1,\dots,a_N r_N}\right)_{r\in(0,\infty)^N} \stackrel{\text{law}}{=} \operatorname{diag}\left(\prod_{j=1}^{N} a_j^{\Theta_j^{(k)}}\right) (Y_r)_{r\in(0,\infty)^N}.$$

That is, the kth component of Y is multi-self-similar with $\Theta^{(k)} = (\Theta_1^{(k)},\dots,\Theta_N^{(k)})$. When $0 < \Theta_j^{(k)} \leq 1$, this is satisfied by N independent fractional Brownian sheets with Hurst multi-indices $H^{(k)} = \Theta^{(k)}$. Furthermore, the discretized versions of the above-discussed fields obviously satisfy Definition 2.5.

Similarly as the notion of self-similarity, the next introduced Lamperti transformation provides definitions acknowledged in the literature as its special cases.

Definition 2.6. *Let $\Theta \in (S^n_+)^N$, and $X = (X_t)_{t\in\mathbb{Z}^N}$ and $Y = (Y_{e^t})_{t\in\mathbb{Z}^N}$. We set*

$$(\mathcal{L}_\Theta X)_{e^t} = e^{t*\Theta} X_t, \quad t \in \mathbb{Z}^N$$
$$(\mathcal{L}_\Theta^{-1} Y)_t = e^{-t*\Theta} Y_{e^t}, \quad t \in \mathbb{Z}^N. \tag{2}$$

The transformation $\mathcal{L}_\Theta$ gives a one-to-one connection between stationary fields and Θ-self-similar fields.

Theorem 2.7. *Let $\Theta \in (S^n_+)^N_c$. If X is stationary, then $\mathcal{L}_\Theta X$ is Θ-self-similar. Conversely, if Y is Θ-self-similar, then $\mathcal{L}_\Theta^{-1} Y$ is stationary. Moreover,*

$$(\mathcal{L}_\Theta^{-1} \circ \mathcal{L}_\Theta)(X) = X \quad \text{and} \quad (\mathcal{L}_\Theta \circ \mathcal{L}_\Theta^{-1})(Y) = Y$$

for all stationary X and Θ-self-similar Y.

Remark 2.8. The proof of Theorem 2.7 reveals the importance of the assumption that the elements of Θ commute pairwisely with respect to the standard matrix product. For $\mathcal{L}_\Theta$ to be a mapping between stationary and Θ-self-similar fields, it is imperative that the identity

$$e^{t*\Theta+s*\Theta} = e^{\sum_{j=1}^N t_j\Theta_j + \sum_{j=1}^N s_j\Theta_j} = e^{\sum_{j=1}^N s_j\Theta_j} e^{\sum_{j=1}^N t_j\Theta_j} = e^{s*\Theta} e^{t*\Theta} \tag{3}$$

holds true for $t, s \in \mathbb{Z}^N$. In general, $e^{A+B} \neq e^A e^B$ for matrix exponents unless A and B commute. Thus, the assumption $\Theta \in (S^n_+)^N_c$ ensures that (3) is valid. In the proofs of later results we need similar identities as (3). Again, these identities follow from our standing assumption.

Next, we turn to stationary increment fields. To this end, we first provide the notions of rectangular and unit cube increments, the latter being a special case of the former. The names of the two types of increments refer to their geometry in the index space.

Definition 2.9. *Let $t, s \in \mathbb{Z}^N$ and $X = (X_t)_{t\in\mathbb{Z}^N}$. The rectangular increment of X corresponding to the hyperrectangle $[s, t]$ is*

$$\begin{aligned}\Delta_{s,t} X &= \sum_{(i_1,\dots,i_N)\in\{0,1\}^N} (-1)^{\sum_{l=1}^N i_l} X_{t_1 - i_1(t_1-s_1),\dots,t_N - i_N(t_N - s_N)} \\ &= \sum_{i\in\{0,1\}^N} (-1)^{\sum i} X_{t-i(t-s)}.\end{aligned} \tag{4}$$

Similarly, for $Y = (Y_{e^t})_{t\in\mathbb{Z}^N}$, we define

$$\begin{aligned}\Delta_{s,t} Y &= \sum_{(i_1,\dots,i_N)\in\{0,1\}^N} (-1)^{\sum_{l=1}^N i_l} Y_{e^{t_1 - i_1(t_1-s_1)},\dots,e^{t_N - i_N(t_N - s_N)}} \\ &= \sum_{i\in\{0,1\}^N} (-1)^{\sum i} Y_{e^{t-i(t-s)}}.\end{aligned}$$

Definition 2.10. *Let $t \in \mathbb{Z}^N$ and $X = (X_t)_{t\in\mathbb{Z}^N}$. The unit cube increment of X at t is given by*

$$\Delta_t X = \Delta_{t-1,t} X = \sum_{(i_1,\ldots,i_N)\in\{0,1\}^N} (-1)^{\sum_{l=1}^N i_l} X_{t_1-i_1,\ldots,t_N-i_N}$$
$$= \sum_{i\in\{0,1\}^N} (-1)^{\sum i} X_{t-i}.$$

Similarly, for $Y = (Y_{e^t})_{t\in\mathbb{Z}^N}$,

$$\Delta_t Y = \Delta_{t-1,t} Y = \sum_{i\in\{0,1\}^N} (-1)^{\sum i} Y_{e^{t-i}}.$$

Remark 2.11. When $N = 1$, the rectangular increment $\Delta_{s,t}X$ is just the usual one-dimensional increment $X_t - X_s$. Due to the alternating sign in (4), degenerate increments ($t_l = s_l$ for some l) are equal to zero. E.g. in Makogin and Mishura[18] the authors considered rectangular increments with $s \le t$ element-wise. However, it can be verified from (4) that the multidimensional increments satisfy yet another property that is analogous with the one-dimensional increments: Let $\tilde{s}$ and $\tilde{t}$ be the vectors obtained from s and t by swapping the elements for which $s_l > t_l$, and let m be the number of such elements. Thus, now $\tilde{s} \le \tilde{t}$ and in addition,

$$\Delta_{s,t}X = (-1)^m \Delta_{\tilde{s},\tilde{t}} X.$$

Example 2.12. In the two-parameter case, we obtain for example that

$$\Delta_{s,t}X = X_{t_1,t_2} - X_{s_1,t_2} - X_{t_1,s_2} + X_{s_1,s_2}$$

and

$$\Delta_t Y = Y_{e^{t_1},e^{t_2}} - Y_{e^{t_1},e^{t_2-1}} - Y_{e^{t_1-1},e^{t_2}} + Y_{e^{t_1-1},e^{t_2-1}}.$$

The following lemma shows how rectangular increments can be constructed from unit square increments.

Lemma 2.13. Let $t, s \in \mathbb{Z}^N$ with $s \le t$ element-wise. Then

$$\sum_{j_1=s_1+1}^{t_1} \cdots \sum_{j_N=s_N+1}^{t_N} \Delta_j Z =: \sum_{j=s+\mathbf{1}}^{t} \Delta_j Z = \Delta_{s,t} Z,$$

where Z is a field indexed by $\mathbb{Z}^N$ and sums of the type $\sum_{j_k=t_k+1}^{t_k}$ are empty sums.

By saying that $(X_t)_{t\in\mathbb{Z}^N}$ is a stationary increment field we simply mean that the unit cube increment field $(\Delta_t X)_{t\in\mathbb{Z}^N}$ is stationary.

Definition 2.14. *A field $X = (X_t)_{t\in\mathbb{Z}^N}$ is a stationary increment field if*

$$(\Delta_{t+s}X)_{t\in\mathbb{Z}^N} \overset{law}{=} (\Delta_t X)_{t\in\mathbb{Z}^N}$$

for every $s \in \mathbb{Z}^N$.

Remark 2.15. In Makogin and Mishura[18] the authors consider several notions of stationarity of increments. In particularly, their notion of strictly stationary rectangular increments corresponds to invariance of joint probability distributions of finite collections of rectangular increments under uniform translations in the parameter space. In our setting, this can be formalized as

$$\begin{pmatrix} \Delta_{s^{(1)}+h,t^{(1)}+h}X \\ \vdots \\ \Delta_{s^{(m)}+h,t^{(m)}+h}X \end{pmatrix} \overset{\text{law}}{=} \begin{pmatrix} \Delta_{s^{(1)},t^{(1)}}X \\ \vdots \\ \Delta_{s^{(m)},t^{(m)}}X \end{pmatrix},$$

where $m \in \mathbb{N}$, $h \in \mathbb{Z}^N$ and $t^{(j)}, s^{(j)} \in \mathbb{Z}^N$ for all $j = 1, \dots, m$. However, since unit cubes provide basic building blocks for our discrete parameter space in the fashion of Lemma 2.13, the above is equivalent with Definition 2.14.

A (discrete) fractional Brownian sheet provides an example of a stationary increment field. In addition, unlike in the case of processes and fractional Brownian motion, fBs is not the only Gaussian self-similar field possessing stationary increments, as the example provided in Makogin and Mishura[17] reveals.

Stationary increment fields satisfying a rather mild additional property form the third essential class of multivariate fields, besides of stationary fields and self-similar fields, considered in this article. We show that the existence of a certain logarithmic moment is a sufficient condition and hence, the class includes e.g., all Gaussian stationary increment fields.

Definition 2.16. *Let $\Theta \in (S_+^n)_c^N$ and let $G = (G_t)_{t\in\mathbb{Z}^N}$ be a stationary increment field. If*

$$\lim_{M_1\to\infty} \cdots \lim_{M_N\to\infty} \sum_{j_1=-M_1}^{t_1} \cdots \sum_{j_N=-M_N}^{t_N} e^{j*\Theta}\Delta_j G \tag{5}$$

converges in probability for every $t \in \mathbb{Z}^N$, then we write $G \in \mathcal{G}_\Theta$. In this case, we denote the limit of (5) as

$$\sum_{j_1=-\infty}^{t_1} \cdots \sum_{j_N=-\infty}^{t_N} e^{j*\Theta}\Delta_j G = \sum_{j=-\infty}^{t} e^{j*\Theta}\Delta_j G. \tag{6}$$

Moreover, if $G_t = 0$ (a.s.) whenever $t_l = 0$ for some $l \in \{1, \dots, N\}$, then we write $G \in \mathcal{G}_{\Theta,0} \subset \mathcal{G}_\Theta$.

Remark 2.17. It turns out that if (5) exists for any permutation of limits, then it exists for all permutation of limits and as a N-fold limit with a common limiting random vector (see also Corollary 2.33). This justifies the abbreviated notation (6).

Lemma 2.18. Let $G = (G_t)_{t\in\mathbb{Z}^N}$ be a stationary increment field. Assume that

$$\mathbb{E}\left(\ln \|\Delta_{\mathbf{1}} G\| \mathbb{1}_{\{\|\Delta_{\mathbf{1}} G\|>1\}}\right)^{N+\delta} < \infty \tag{7}$$

for some $\delta > 0$, where $\Delta_{\mathbf{1}} = \Delta_{1,\dots,1}$. Then (5) converges almost surely for all $\Theta \in (S_+^n)_c^N$. In particularly, $G \in \mathcal{G}_\Theta$.

Moreover, if $G_t = 0$ whenever $t_l = 0$ for some $l \in \{1,\dots,N\}$, then $\|\Delta_{\mathbf{1}} G\|$ can be replaced by $\|G_{\mathbf{1}}\| = \|G_{1,\dots,1}\|$ in (7). In this case, $G \in \mathcal{G}_{\Theta,0}$ for all $\Theta \in (S_+^n)_c^N$.

Corollary 2.19. *Let $G = (G_t)_{t\in\mathbb{Z}^N}$ be a stationary increment field. If*

$$\mathbb{E}\|\Delta_{\mathbf{1}} G\| < \infty,$$

then $G \in \mathcal{G}_\Theta$ for all $\Theta \in (S_+^n)_c^N$. In particularly, this holds if $\mathbb{E}\|G_t\| < \infty$ for all $t \in \mathbb{Z}^N$.

Next, we define a transformation $\mathcal{M}_\Theta$ giving a one-to-one connection between Θ-self-similar fields and fields in $\mathcal{G}_{\Theta,0}$. The definition of the inverse $\mathcal{M}_\Theta^{-1}$ involving infinite sums may be regarded as a formal at this point.

Definition 2.20. *Let $\Theta \in (S_+^n)_c^N$, and $G = (G_t)_{t\in\mathbb{Z}^N}$ and $Y = (Y_{e^t})_{t\in\mathbb{Z}^N}$. We define*

$$(\mathcal{M}_\Theta Y)_t = (-1)^{|-u|} \sum_{j_u=\mathbf{1}}^{t_u} \sum_{j_{-u}=t_{-u}+\mathbf{1}}^{\mathbf{0}} e^{-j*\Theta} \Delta_j Y$$

and

$$(\mathcal{M}_\Theta^{-1} G)_{e^t} = \sum_{j=-\infty}^{t} e^{j*\Theta} \Delta_j G. \tag{8}$$

The following theorem provides the second main result of this article.

Theorem 2.21. *If Y is Θ-self-similar, then $\mathcal{M}_\Theta Y \in \mathcal{G}_{\Theta,0}$. If $G \in \mathcal{G}_\Theta$, then $\mathcal{M}_\Theta^{-1} G$ is Θ-self-similar. Moreover,*

$$(\mathcal{M}_\Theta^{-1} \circ \mathcal{M}_\Theta)(Y) = Y \quad \text{and} \quad (\mathcal{M}_\Theta \circ \mathcal{M}_\Theta^{-1})(G) = G$$

for all Θ-self-similar Y and $G \in \mathcal{G}_{\Theta,0}$.

Corollary 2.22. *If X is stationary, then $(\mathcal{M}_\Theta \circ \mathcal{L}_\Theta)(X) \in \mathcal{G}_{\Theta,0}$. If $G \in \mathcal{G}_\Theta$, then $(\mathcal{L}_\Theta^{-1} \circ \mathcal{M}_\Theta^{-1})(G)$ is stationary. Moreover, $\mathcal{M}_\Theta \circ \mathcal{L}_\Theta$ is a bijection between stationary fields and fields in $\mathcal{G}_{\Theta,0}$.*

Remark 2.23. By carefully examining the proof of Theorem 2.21, the following observations can be made. Let P be a predicate and let $\mathcal{G}_{\Theta,P} \subset \mathcal{G}_{\Theta}$ consist of the fields satisfying P. Assume that the definition of $\mathcal{M}_{\Theta}$ is modified in such a way that $\Delta_t \mathcal{M}_{\Theta} Y = e^{-t*\Theta} \Delta_t Y$ still holds and $\mathcal{M}_{\Theta} Y \in \mathcal{G}_{\Theta,P}$. Now,

$$(\mathcal{M}_{\Theta}^{-1} \circ \mathcal{M}_{\Theta})(Y) = Y$$

for all self-similar Y. Moreover, if P together with fixed increments define a unique field in $\mathcal{G}_{\Theta,P}$, then

$$(\mathcal{M}_{\Theta} \circ \mathcal{M}_{\Theta}^{-1})(G) = G$$

for all $G \in \mathcal{G}_{\Theta,P}$. That is, in Theorem 2.21 we obtain a bijection between the sets of Θ-self-similar fields and $\mathcal{G}_{\Theta,P}$. See also Remark 2.35, where examples of such sets $\mathcal{G}_{\Theta,P}$ are discussed.

Furthermore, the remaining theorems hold true for such modified transformations $\mathcal{M}_{\Theta}$ when the class $\mathcal{G}_{\Theta,0}$ is replaced with the appropriate $\mathcal{G}_{\Theta,P}$.

Last, we examine solutions of equations that can be interpreted as generalized AR(1) equations, where the noise field is of class $\mathcal{G}_{\Theta}$. To this end, we first define the "AR(1) part" that consists of random vectors that are previous at least in one coordinate direction of the parameter space.

Definition 2.24. *Let $\Theta \in (S_+^n)^N$ and $X = (X_t)_{t \in \mathbb{Z}^N}$. Then, for every $i \in \{0,1\}^N$ with $i \neq \mathbf{0}$, we define*

$$\hat{\Theta}_i = (-1)^{1+\sum_{l=1}^N i_l} e^{-i*\Theta} \quad \text{and} \quad \hat{X}_{t,i} = X_{t_1-i_1,\ldots,t_N-i_N}, \quad t \in \mathbb{Z}^N.$$

Furthermore, let $\hat{\Theta} = (\hat{\Theta}_i) \in (S^n)^{2^N-1}$ and $\hat{X}_t = (\hat{X}_{t,i}) \in (\mathbb{R}^n)^{2^N-1}$, where the elements of $\hat{\Theta}$ and $\hat{X}_t$ are ordered consistently. Now

$$\hat{\Theta} \star \hat{X}_t = \sum_{\substack{(i_1,\ldots,i_N)\in\{0,1\}^N \\ i \neq \mathbf{0}}} (-1)^{1+\sum_{l=1}^N i_l} e^{-i*\Theta} X_{t_1-i_1,\ldots,t_N-i_N}.$$

Example 2.25. The cases $N = 1$ and $N = 2$ yield

$$\hat{\Theta} \star \hat{X}_t = e^{-\Theta} X_{t-1}$$

and

$$\hat{\Theta} \star \hat{X}_t = e^{-\Theta_1} X_{t_1-1,t_2} + e^{-\Theta_2} X_{t_1,t_2-1} - e^{-\Theta_1-\Theta_2} X_{t_1-1,t_2-1},$$

respectively.

The proofs of the next two theorems follow closely the lines of the corresponding proofs for univariate fields in Voutilainen et al.[26]. However, for the reader's convenience, they are presented at the end of Section 4. The first theorem shows that generalized AR(1) equations admit stationary solutions, whereas the second one shows that stationary fields solve this type of equations. The connection between the stationary solution and the noise field is given by the composition $\mathcal{M}_\Theta \circ \mathcal{L}_\Theta$.

Theorem 2.26. *Let $\Theta \in (S^n_+)^N_c$ and $X = (X_t)_{t\in\mathbb{Z}^N}$. If*

(i)

$$\lim_{m\to-\infty} e^{m\Theta_j} X_{t_1,\ldots,t_{j-1},m,t_{j+1},\ldots,t_N} \xrightarrow{\mathbb{P}} 0$$

for every $j \in \{1,\ldots,N\}$ and $t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_N \in \mathbb{Z}$.

(ii) For $G \in \mathcal{G}_\Theta$,

$$X_t = \hat{\Theta} \star \hat{X}_t + \Delta_t G, \quad t \in \mathbb{Z}^N,$$

then X is stationary with $X = (\mathcal{L}_\Theta^{-1} \circ \mathcal{M}_\Theta^{-1})(G)$.

Remark 2.27. $X = (\mathcal{L}_\Theta^{-1} \circ \mathcal{M}_\Theta^{-1})(G)$ is the unique stationary solution to

$$X_t = \hat{\Theta} \star \hat{X}_t + \Delta_t G, \quad t \in \mathbb{Z}^N.$$

Theorem 2.28. *Let $\Theta \in (S^n_+)^N_c$ and let $X = (X_t)_{t\in\mathbb{Z}^N}$ be stationary. Set $G = (\mathcal{M}_\Theta \circ \mathcal{L}_\Theta)(X)$. Then*

$$X_t = \hat{\Theta} \star \hat{X}_t + \Delta_t G \quad \text{for all } t \in \mathbb{Z}^N. \tag{9}$$

The last theorem characterizes stationary fields as solutions to (9), where the noise field belongs to $\mathcal{G}_{\Theta,0}$.

Theorem 2.29. *Let $\Theta \in (S^n_+)^N_c$. Then $X = (X_t)_{t\in\mathbb{Z}^N}$ is stationary if and only if*

(i)

$$\lim_{m\to-\infty} e^{m\Theta_j} X_{t_1,\ldots,t_{j-1},m,t_{j+1},\ldots,t_N} \xrightarrow{\mathbb{P}} 0$$

for every $j \in \{1,\ldots,N\}$ and $t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_N \in \mathbb{Z}$.

(ii)

$$X_t = \hat{\Theta} \star \hat{X}_t + \Delta_t G, \quad t \in \mathbb{Z}^N$$

with $G \in \mathcal{G}_{\Theta,0}$.

Moreover, the fields are connected as $X = (\mathcal{L}_\Theta^{-1} \circ \mathcal{M}_\Theta^{-1})(G)$ and $G = (\mathcal{M}_\Theta \circ \mathcal{L}_\Theta)(X)$.

2.2.1. Auxiliary results of Theorem 2.21

Here, we present results that lead to Theorem 2.21, but they may also be interesting by their own merit. The first one shows that if $G \in \mathcal{G}_\Theta$, then (5) defines a Θ-self-similar field.

Lemma 2.30. Let $\Theta \in (S_+^n)_c^N$ and $G \in \mathcal{G}_\Theta$. Define

$$Y_{e^t} = \sum_{j=-\infty}^{t} e^{j*\Theta} \Delta_j G, \tag{10}$$

then $\Delta_t Y = e^{t*\Theta} \Delta_t G$ for all $t \in \mathbb{Z}^N$. Moreover, Y is Θ-self-similar.

Remark 2.31. In the proof of the first claim of 2.30, the assumptions $\Theta \in (S_+^n)_c^N$ and $G \in \mathcal{G}_\Theta$ are used only to ensure that Y_{e^t} defined by (10) makes sense as a random vector. If we regard (10) as a formal series, then these assumptions can be dropped still obtaining $\Delta_t Y = e^{t*\Theta} \Delta_t G$ for all $t \in \mathbb{Z}^N$.

The combination of the next two lemmas gives us means to construct fields belonging to $\mathcal{G}_\Theta$ from Θ-self-similar fields.

Lemma 2.32. Let $\Theta \in (S_+^n)^N$ and Y be Θ-self-similar. If for $G = (G_t)_{t \in \mathbb{Z}^N}$ it holds that

$$\Delta_t G = e^{-t*\Theta} \Delta_t Y, \quad t \in \mathbb{Z}^N,$$

then

$$\sum_{j=-\infty}^{t} e^{j*\Theta} \Delta_j G = Y_{e^t}.$$

Moreover, if $\Theta \in (S_+^n)_c^N$, then G is stationary increment field and thus $G \in \mathcal{G}_\Theta$.

Corollary 2.33. *Assume that $G \in \mathcal{G}_\Theta$ according to Definition 2.16. By Lemma 2.30 the limiting field is Θ-self-similar with $\Delta_t Y = e^{t*\Theta} \Delta_t G$. By multiplying with the inverse matrix $e^{-t*\Theta}$ from the left, we obtain $\Delta_t G = e^{-t*\Theta} \Delta_t Y$. Now the proof of Lemma 2.32 shows that*

$$\lim_{M \to \infty} \sum_{j=-M}^{t} e^{j*\Theta} \Delta_j G = Y_{e^t}$$

regardless of how we approach the limit. This verifies the claim of Remark 2.17.

In (11), we apply the notation introduced at the beginning of Subsection 2.1.

Lemma 2.34. Let $t \in \mathbb{Z}^N$ and let $u \subseteq \{1, \dots, N\}$ be the set of indices l for which $t_l \geq 0$. Moreover, let Y be a field indexed by $\mathbb{Z}^N$. We define

$$G_t = (-1)^{|-u|} \sum_{j_u = \mathbf{1}}^{t_u} \sum_{j_{-u} = t_{-u} + \mathbf{1}}^{\mathbf{0}} e^{-j*\Theta} \Delta_j Y, \tag{11}$$

where $j = (j_1, \dots, j_N)$ is a multi-index and Θ is an arbitrary N-tuple of $n \times n$-matrices. Now

$$\Delta_t G = e^{-t*\Theta}\Delta_t Y \quad \text{for all } t \in \mathbb{Z}^N$$

and $G_t = 0$ for all t such that $t_l = 0$ for some $l \in \{1, \ldots, N\}$.

Remark 2.35. Definition (11) can straightforwardly be modified in such a way that $\Delta_t G = e^{-t*\Theta}\Delta_t Y$ still holds and G_t vanishes on hyperplanes $t \in \mathbb{Z}^N$ with $t_l = c_l$ for some l, and $c_l \in \mathbb{R}$. In addition, if we set $\tilde{G} = G + \xi$, where ξ is a random variable on the underlying probability space, we obtain that $\Delta_t \tilde{G} = e^{-t*\Theta}\Delta_t Y$ and $\tilde{G}_t = \xi$ for $t \in \mathbb{Z}^N$ such that $t_l = c_l$ for some l. See also Voutilainen et al.[26], where G_t defined from a self-similar Y with $\Delta_t G = e^{-t*\Theta}\Delta_t Y$ vanishes on hyperplanes $\sum_{l=1}^N t_l \in \{0, -1, \ldots, -N+1\}$. Thus, under $\Theta \in (S_+^n)_c^N$ and by Lemma 2.32, we obtain stationary increment fields belonging to different subsets of $\mathcal{G}_\Theta$.

2.3. Multivariate stationary fractional Ornstein-Uhlenbeck fields

Let A be a $n \times n$-matrix and let $B_t = (B_t^{H^{(1)}}, \ldots B_t^{H^{(n)}})_{t\in\mathbb{Z}^N}$ consist of independent fractional Brownian sheets with Hurst indices $H^{(k)} = (H_1^{(k)}, \ldots, H_N^{(k)})$, $0 < H_j^{(k)} \le 1$. Set $G_t = AB_t$. Then $\Delta_t G = A\Delta_t B$ and G is clearly a stationary increment field, and since G is Gaussian, it belongs to $\mathcal{G}_\Theta$ for every $\Theta \in (S_+^n)_c^N$ by Lemma 2.18. This allows us to define (discrete) stationary fractional Ornstein-Uhlenbeck fields of the first kind as

$$X = (\mathcal{L}_\Theta^{-1} \circ \mathcal{M}_\Theta^{-1})(G).$$

Set $\Theta = (\Theta_1, \ldots, \Theta_N)$, where $\Theta_j = \operatorname{diag}(H_j^{(k)})$. In the light of the discussion after Definition 2.5,

$$(B_{e^{t+s}})_{t\in\mathbb{Z}^N} \overset{law}{=} \operatorname{diag}\left(\prod_{j=1}^N e^{s_j H_j^{(k)}}\right)(B_{e^t})_{t\in\mathbb{Z}^N} = e^{s*\Theta}(B_{e^t})_{t\in\mathbb{Z}^N}.$$

That is, B is Θ-self-similar. Furthermore, set $Y_{e^t} = AB_{e^t}$. Then, under the assumption that A and $e^{s*\Theta}$ commute,

$$\begin{aligned}(Y_{e^{t+s}})_{t\in\mathbb{Z}^N} &= A(B_{e^{t+s}})_{t\in\mathbb{Z}^N} \overset{law}{=} Ae^{s*\Theta}(B_{e^t})_{t\in\mathbb{Z}^N} \\ &= e^{s*\Theta}(AB_{e^t})_{t\in\mathbb{Z}^N} = e^{s*\Theta}(Y_{e^t})_{t\in\mathbb{Z}^N}.\end{aligned}$$

showing that also Y is Θ-self-similar. Note that this holds e.g. when A is diagonal or $H^{(k)} = H = (H_1, \ldots, H_N)$, since in the latter case

$$e^{s*\Theta} = \operatorname{diag}\left(\prod_{j=1}^N e^{s_j H_j}\right) = I\prod_{j=1}^N e^{s_j H_j}.$$

This approach allows us to define (discrete) stationary fractional Ornstein-Uhlenbeck fields of the second kind as

$$X = \mathcal{L}_\Theta^{-1} Y.$$

Last, we point out that the above can be applied also e.g. to the Gaussian self-similar field with stationary increments that is not a fBs introduced in Makogin and Mishura[17]. In this way, we are able to construct other stationary Gaussian fractional Ornstein-Uhlenbeck fields of the first and second kind.

3. Applications and simulations

We cover some of the potential applications related to our research in Subsection 3.1. Subsection 3.2 presents a simulation study of bivariate Ornstein-Uhlenbeck fields. A visualization of the discussed estimation method of the self-similarity parameter connects the two subsections together.

3.1. On applications

Historically, the urge for the study of self-similarity has arisen especially in the geosciences as fractals are found in various natural phenomena and patterns[19, 20]. Applications of self-similarity can also be found in finance, physics, telecommunications, and materials sciences, among others (see e.g., Lee et al. [14, 15, 24] and references therein). In addition, stationary random objects with distinct dependency properties can be utilized in description of many subjects of interest in geostatistics, and in numerous other scientific disciplines. For details on random fields in geostatistics with real-world examples, we refer the reader to Cressie[4].

When applied to modeling, the parameter spaces of random fields represent typically the spatial locations of quantities that are interpreted as realizations of random variables or vectors, see e.g., Cressie[4, 21]. Although the underlying random field may be considered to be continuous, the observations are discrete corresponding to a realization of a discretization of the field. Moreover, in contrast to the traditional definition of self-similarity Genton et al.[8], the discussed multi-self-similarity takes account of different fractal behavior in different coordinate directions, which is desirable in several modeling problems[14, 22].

The introduced bijective transformations allow transitioning between the classes of stationary, self-similar, and stationary increment random fields, and therefore applying a variety of tools when the goal is to analyze or model a particular field of these types. For example, one can utilize the machinery of stationary fields in the study of a supposed self-similar field via the transformation $\mathcal{L}_{\Theta}^{-1}$. This approach was taken in Lee et al.[14], where the authors proposed a statistical test for the multi-self-similarity property of a univariate random field Y, and in the case of the approval of the hypothesis provided estimates for the vector-valued self-similarity parameter. The method is based on testing the stationarity of the inverse Lamperti transformation $X = \mathcal{L}_{\Theta}^{-1} Y$. More details such as different statistical tests for stationarity can be found in Lee et al.[14]. Although some computational challenges may arise, the method could be extended to cover also multivariate fields discussed in this article. We illustrate this idea in the end of the next subsection. Last, we mention that when desired, one could as well study the stationarity of the increments of $G = \mathcal{M}_\Theta Y$ that are of the form $\Delta_t G = e^{-t*\Theta}\Delta_t Y$ by Lemma 2.32.

3.2. Simulations of bivariate stationary Ornstein-Uhlenbeck sheets

In this subsection we present simulation results of the fields obtained from a bivariate Brownian sheet via transformations $\mathcal{L}_\Theta^{-1} \circ \mathcal{M}_\Theta^{-1}$ and $\mathcal{L}_\Theta^{-1}$. The code is available as supplementary material[1]. The transformations are applied to the standard bivariate Brownian sheet $B = (B^{(1)}, B^{(2)})$, where $B^{(1)}$ and $B^{(2)}$ are independent copies of the Brownian sheet. Hence, B is Θ-self-similar with $\Theta = (\Theta_1, \Theta_2)$, where $\Theta_i = \text{diag}(\frac{1}{2}, \frac{1}{2})$. The self-similarity parameter dictates the parameter in $X^{OU2} := \mathcal{L}_\Theta^{-1}(B)$, and the same parameter is also used in $X^{OU1} := (\mathcal{L}_\Theta^{-1} \circ \mathcal{M}_\Theta^{-1})(B)$. The resulting fields are known as stationary Ornstein-Uhlenbeck sheets of the second and the first kind, respectively, and they both are centered and Gaussian. It can be shown directly from the definitions of the involved transformations that X^{OU2} has unit variance and the variance of X^{OU1} is approximately 2.5.

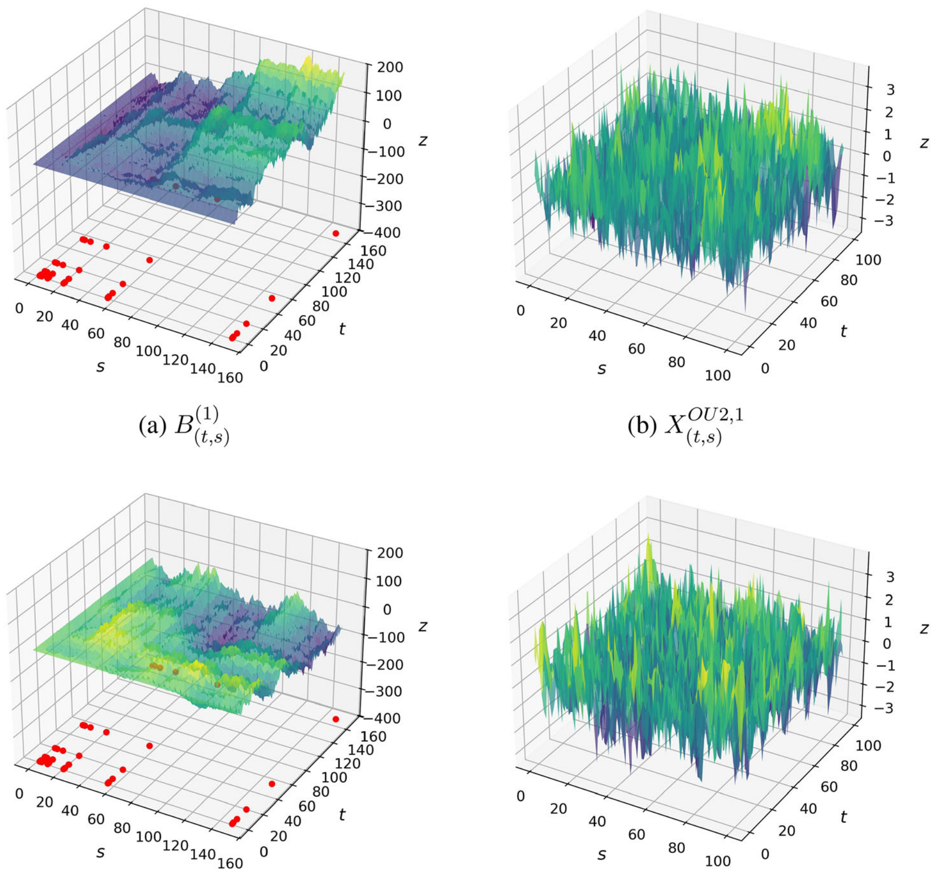
We start by demonstrating the results for the transformation $\mathcal{L}_\Theta^{-1}$. To this end, we simulate the bivariate Brownian sheet B described above over the discrete two-dimensional time grid $\{(t,s)|t,s \in \{e^0, e^1, ..., e^{99}, e^{100}\}\}$ using addition properties similar to Lemma 2.13 and independence of increments together with the fact that

$$B^{(i)}_{t+\Delta t, s+\Delta s} - B^{(i)}_{t+\Delta t, s} - B^{(i)}_{t, s+\Delta s} + B^{(i)}_{t,s} \sim \mathcal{N}(0, \Delta t \times \Delta s)$$

for $i = 1, 2$. This results in a total of 10201 $(= 101 \times 101)$ observations for the Brownian sheet in both spatial dimensions. After this, we apply the $\mathcal{L}_\Theta^{-1}$ transformation to B to obtain the stationary Ornstein-Uhlenbeck field of the second kind over the grid $\{(t,s)|t,s \in \{0, 1, ..., 100\}\}$. Since we use "exponential clocks" in the simulations, the variance of the Brownian sheet will explode. Consequently, it becomes infeasible to visualize the sheet with the actual grid that we use as the basis for the transformation, and we instead plot the sheet for the time grid $\{(t,s)|t,s \in \{0, 1, ..., 150\}\}$ and mark the points on our actual grid that fall in $[0, 150]^2$. Figure 1 shows the results. The panels on the left show the Brownian sheet and the panels on the right show the stationary field resulting from the transformation in both spatial dimensions separately.

Next, we illustrate the transformation $\mathcal{L}_\Theta^{-1} \circ \mathcal{M}_\Theta^{-1}$. We first simulate $(N + 101) \times (N + 101)$ Brownian increments over the two-dimensional time grid

[1]The core functionality of the code was written manually, but we used Chat GPT (versions GPT-4o and o1 depending on the task) to detect errors, refine and clean up the code, and optimize the performance of specific functions.



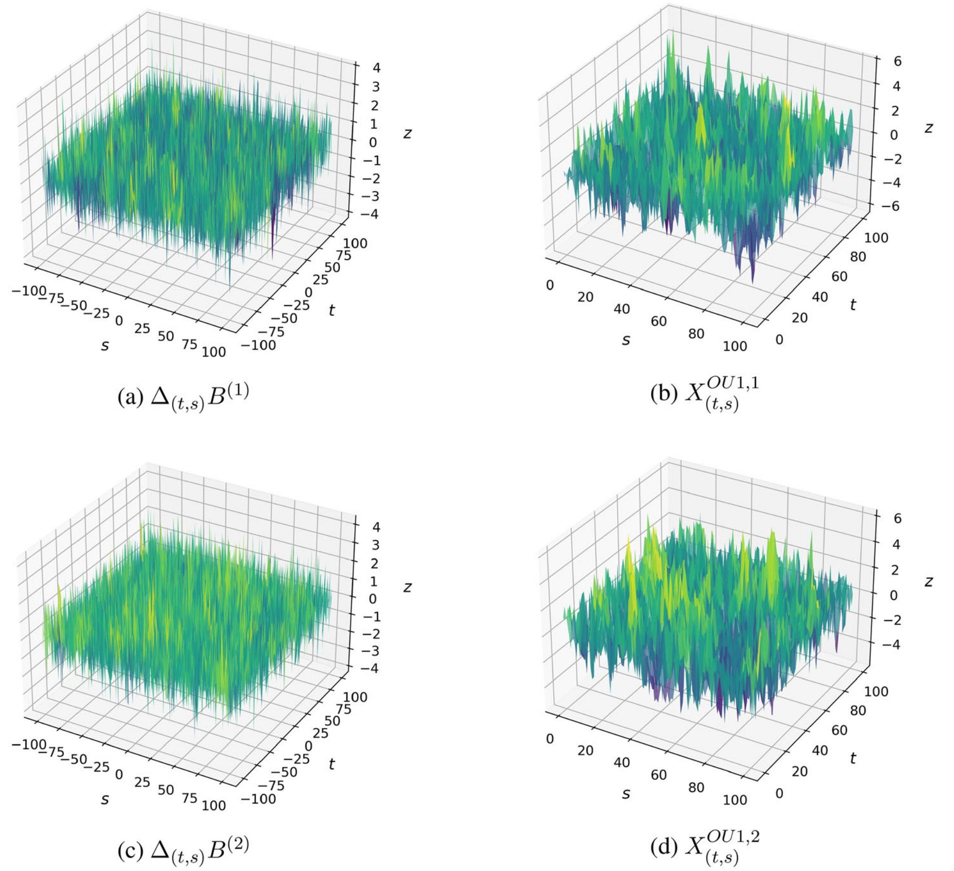
(a) $B^{(1)}_{(t,s)}$ (b) $X^{OU2,1}_{(t,s)}$

(c) $B^{(2)}_{(t,s)}$ (d) $X^{OU2,2}_{(t,s)}$

Figure 1. $\mathcal{L}_{\Theta}^{-1}$ transformation**.** Panels (a) and (c) show the first and the second spatial dimension of the bivariate Brownian sheet over the grid $\{(t,s)|s,t \in \{0,1,2,...,150\}\}$. The red points indicate the locations of the exponential grid $\{(t,s)|s,t \in \{e^0,e^1,...,e^{100}\}\}$ within the used range. The transformation is applied to the Brownian sheet evaluated on the exponential grid (see Equation (2)), but because the gaps between the grid points grow exponentially, the variance also explodes, making it challenging to visualize the sheet for the entire exponential grid. Hence, the figure only demonstrates a part of the sheet for the regular time domain with unit increments. Panel (b) and (d) present the spatial dimensions of the stationary Ornstein-Uhlenbeck field of the second kind resulting from the transformation.

$\{(t,s)|t,s \in \{-N,-(N-1),...,99,100\}\}$ for both spatial dimensions. Since the sum in Equation (8) runs from $-\infty$ to t, we have to decide an arbitrary cutoff point. In our experiments, we use $N = 100$ and fix the size of the window over which we take the sums to $N \times N$, so that we start the computations at $t,s = 0$ with the specified window size. Therefore, at $t,s = 0$, the window covers the whole negative orthant of the time domain and moves accordingly as we increase t and s.

Figure 2 shows the results. The panels on the left show the Brownian increments on $[-100,100]^2$ used for the transformation and the panels on the right



(a) $\Delta_{(t,s)}B^{(1)}$

(b) $X^{OU1,1}_{(t,s)}$

(c) $\Delta_{(t,s)}B^{(2)}$

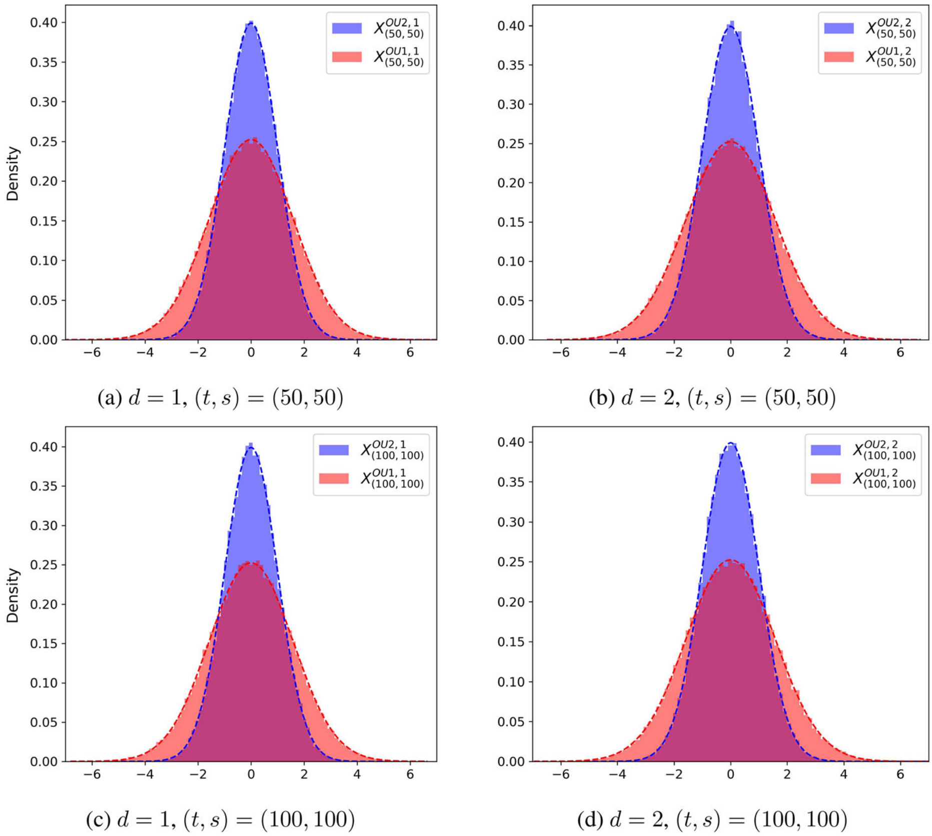
(d) $X^{OU1,2}_{(t,s)}$

Figure 2. $\mathcal{L}_{\Theta}^{-1} \circ \mathcal{M}_{\Theta}^{-1}$ **transformation.** The figure visualizes the $\mathcal{L}_{\Theta}^{-1} \circ \mathcal{M}_{\Theta}^{-1}$ transformation applied to the bivariate Brownian field. Panels (a) and (c) show the first and second spatial dimensions of the stationary increments of the Brownian field. Panel (b) and (d) present the first and second spatial dimensions of the stationary Ornstein-Uhlenbeck field of the first kind resulting from the transformation.

show the stationary Ornstein-Uhlenbeck field of the first kind in both spatial dimensions. As with the Brownian sheet, visualizing the middle step, the $\mathcal{M}_{\Theta}^{-1}$ transformation, is not feasible due to the exploding variance, and hence we only plot the final result, which is defined over $\{(t,s)|t,s \in \{0,1,...,100\}\}$.

Next, we simulate both transformations 10^5 times and illustrate the results with histograms. We compare the distributions of the Ornstein-Uhlenbeck fields at two time points $(t,s) \in \{(100,100),(50,50)\}$ in both spatial dimensions. Figure 3 plots the histograms and Table 1 shows different statistics computed for the transformations. Both of these objects manifest stationarity and normality of the fields, and the difference of the variances between the fields.

Finally, we demonstrate the transformation $\mathcal{L}_{\Theta}^{-1}$ with a slightly perturbed Θ. As mentioned, B is self-similar with $\Theta_i = \text{diag}(\frac{1}{2},\frac{1}{2})$. Here, we demonstrate what



(a) $d = 1$, $(t, s) = (50, 50)$

(b) $d = 2$, $(t, s) = (50, 50)$

(c) $d = 1$, $(t, s) = (100, 100)$

(d) $d = 2$, $(t, s) = (100, 100)$

Figure 3. Empirical distributions of the stationary Ornstein-Uhlenbeck fields**.** The figure shows the empirical distributions at two different points in time, $(t, s) = (50, 50)$ and $(t, s) = (100, 100)$ for both spatial dimensions, d. The dashed blue line is the density function of the normal distribution with a mean of 0 and a variance of 1. Correspondingly, the red line is the density function of the normal distribution with a mean of 0 and a variance of 2.5. The histograms were computed from 10^5 simulations.

Table 1. Statistics for the transformations.

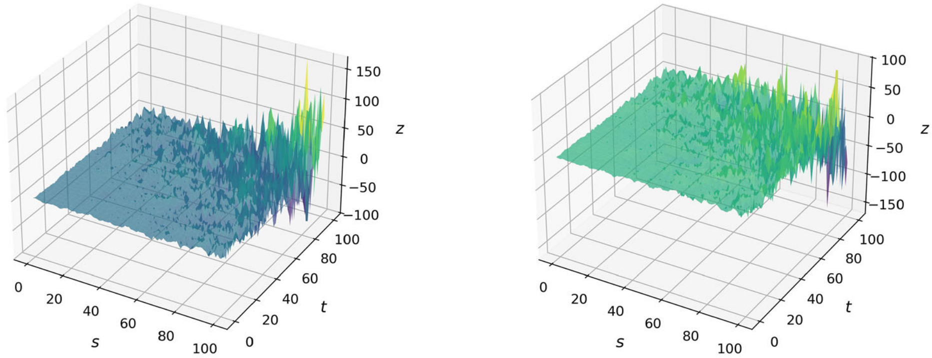
(t, s)	*dim*	$\hat{\mu}_{X^{OU2}}$	$\hat{\sigma}^2_{X^{OU2}}$	$\hat{\mu}_{X^{OU1}}$	$\hat{\sigma}^2_{X^{OU1}}$
(50,50)	1	-0.001	1.000	-0.010	2.509
(50,50)	2	0.000	0.999	-0.006	2.503
(100,100)	1	0.003	1.000	-0.006	2.483
(100,100)	2	0.003	0.996	0.003	2.499

This table shows the sample means and variances computed from the simulation results presented in Figure 3. The tuple (t, s) denotes time, *dim* is the spatial dimension, and $\hat{\mu}$ and $\hat{\sigma}^2$ denote the sample mean and sample variance, respectively.

happens if we use

$$\tilde{\Theta}_i = \begin{pmatrix} \frac{1}{2} & \epsilon \\ \epsilon & \frac{1}{2} \end{pmatrix} \tag{12}$$

instead. We set $\epsilon = \frac{1}{45}$ as the perturbation parameter because it produces clear and visually interpretable results for the grid scales that we use. Figure 4



(a) $\tilde{X}^{(1)}_{(t,s)}$ (b) $\tilde{X}^{(2)}_{(t,s)}$

Figure 4. $\mathcal{L}_\Theta^{-1}$ **transformation with perturbed** Θ**.** This figure shows the results of the $\mathcal{L}_\Theta^{-1}$ transformation if we use the perturbed $\tilde{\Theta} = (\tilde{\Theta}_1, \tilde{\Theta}_2)$ defined in (12) as the parameter of the transformation. The field $\tilde{X}$ exhibits clear signs of non-stationarity.

shows the results. It is immediately apparent that the field produced by the transformation is not stationary, which consequently would lead to rejection of the supposed self-similarity parameter $\tilde{\Theta}$.

4. Proofs and auxiliaries

Proof of Theorem 2.7. Assume that X is stationary. Set $Y_{e^t} = (\mathcal{L}_\Theta X)_{e^t}$. Then, for $s \in \mathbb{Z}^N$,

$$Y_{e^{t+s}} = e^{(t+s)*\Theta} X_{t+s} \overset{\text{law}}{=} e^{t*\Theta + s*\Theta} X_t = e^{s*\Theta} e^{t*\Theta} X_t = e^{s*\Theta} Y_{e^t},$$

since the matrices $t*\Theta$ and $s*\Theta$ commute. Next, assume that Y is Θ-self-similar. Set $X_t = (\mathcal{L}_\Theta^{-1} Y)_t$. Then, for $s \in \mathbb{Z}^N$,

$$X_{t+s} = e^{-(t+s)*\Theta} Y_{e^{t+s}} \overset{\text{law}}{=} e^{-t*\Theta - s*\Theta} e^{s*\Theta} Y_{e^t} = e^{-t*\Theta} Y_{e^t} = X_t$$

again by commutation of the matrix exponents. The finite dimensional distributions can be treated similarly. The fact that $\mathcal{L}_\Theta$ is a bijection follows directly from the definition. □

Next, we state an elementary result for sums of binomial coefficients that we utilize in several occasions. A proof can be found in Voutilainen et al.[26]

Lemma 4.1. Let $M \in \mathbb{N}$. Then

$$\sum_{m=0}^{M} (-1)^m \binom{M}{m} = 0.$$

Proof of Lemma 2.13. If $s_l = t_l$ for some l, then the left hand side involves an empty sum and thus, we get $0 = \Delta_{s,t} Z$ agreeing with Remark 2.11. Assume now that $s < t$ element-wise. Also, for the sake of simplicity, we assume that the parameter space of Z is $\mathbb{Z}^N$. The same proof holds also e.g. for self-similar fields with the parameter space $\{e^t : t \in \mathbb{Z}^N\}$. Now

$$\sum_{j=s+\mathbf{1}}^{t} \Delta_j Z = \sum_{j_1=s_1+1}^{t_1} \cdots \sum_{j_N=s_N+1}^{t_N} \sum_{i\in\{0,1\}^N} (-1)^{\sum i} Z_{j_1-i_1,\dots,j_N-i_N} \tag{13}$$

consists of terms $Z_{k_1,\dots,k_N}$ with $s_l \le k_l \le t_l$ for all l. Let $k_l = t_l$ for all $l \in M_+ \subseteq \{1,\dots,N\}$ and $k_l = s_l$ for all $l \in M_- \subseteq \{1,\dots,N\}$. We denote the cardinality of such sets as $|M_-|$. Now $Z_{k_1,\dots,k_N}$ belongs to (13) if and only if

(i) For all $l \in M_+$: $j_l = t_l$ and $i_l = 0$.
(ii) For all $l \in M_-$: $j_l = s_l$ and $i_l = 1$.
(iii) For all $l \notin M_+ \cup M_-$ either

(a) $j_l = k_l$ and $i_l = 0$
(b) $j_l = k_l + 1$ and $i_l = 1$

Thus, the total number of terms $Z_{k_1,\dots,k_N}$ in (13) is

$$(-1)^{|M_-|}\left[\binom{N-|M_+\cup M_-|}{0} - \binom{N-|M_+\cup M_-|}{1} \pm \binom{N-|M_+\cup M_-|}{N-|M_+\cup M_-|}\right] = 0$$

whenever $|M_+ \cup M_-| \neq N$ by Lemma 4.1. The sign of the last binomial coefficient depends on the parity of $N-|M_+\cup M_-|$. In the case that $M_+\cup M_- = \{1,\dots,N\}$, the number of terms is $(-1)^{|M_-|}$, where $|M_-|$ is the number of indices for which $k_l = s_l$, and $k_l = t_l$ for all the other indices. Hence, (13) reduces to

$$\sum_{j_1=s_1+1}^{t_1} \cdots \sum_{j_N=s_N+1}^{t_N} \sum_{i\in\{0,1\}^N} (-1)^{\sum i} Z_{j_1-i_1,\dots,j_N-i_N} = \sum_{(i_1,\dots,i_N)\in\{0,1\}^N} (-1)^{\sum_{l=1}^N i_l} Z_{t_1-i_1(t_1-s_1),\dots,t_N-i_N(t_N-s_N)} = \Delta_{s,t} Z$$

□

Proof of Lemma 2.18. The latter claim follows directly from the fact that $\Delta_{\mathbf{1}} G = G_{\mathbf{1}}$, if G_t vanishes on the hyperplanes where $t_l = 0$ for some l.

Let $t \in \mathbb{Z}^N$. We begin the proof of the first claim by showing that $\sup_{j\le t} \|e^{j*\Theta}\Delta_j G\| < \infty$ almost surely, where $j \le t$ is understood component-wise. For this, let $k \in \mathbb{N}\cup\{0\}$. We use the abbreviated notation $\sum j = \sum_{l=1}^N j_l$

and $\sum t = \sum_{l=1}^{N} t_l$, where in addition $j_l \leq t_l$ for all l. Now

$$\begin{aligned}\mathbb{P}\left(\sup_{\substack{\sum j=\sum t-k\\ j\leq t}} \|e^{j*\Theta}\Delta_j G\| > \epsilon\right) &= \mathbb{P}\left(\bigcup_{\substack{\sum j=\sum t-k\\ j\leq t}} \left\{\|e^{j*\Theta}\Delta_j G\| > \epsilon\right\}\right)\\ &\leq \sum_{\substack{\sum j=\sum t-k\\ j\leq t}} \mathbb{P}\left(\|e^{j*\Theta}\Delta_j G\| > \epsilon\right)\\ &\leq \sum_{\substack{\sum j=\sum t-k\\ j\leq t}} \mathbb{P}\left(\|e^{j*\Theta}\|\|\Delta_j G\| > \epsilon\right)\\ &= \sum_{\substack{\sum j=\sum t-k\\ j\leq t}} \mathbb{P}\left(\|\Delta_{\mathbf{1}} G\| > \frac{\epsilon}{\|e^{j*\Theta}\|}\right)\end{aligned} \tag{14}$$

by stationarity of increments. We set $u_j = \{l : j_l < 0\}$. Note that when k is large enough, at least one of the indices j_l has to be negative. Now

$$\begin{aligned}\|e^{j*\Theta}\| &= \|\prod_{l=1}^{N} e^{j_l\Theta_l}\| \leq \prod_{l=1}^{N}\|e^{j_l\Theta_l}\| = \prod_{l\in u_j}\|e^{j_l\Theta_l}\|\prod_{l\notin u_j}\|e^{j_l\Theta_l}\| \leq C_u\prod_{l\in u_j}\|e^{j_l\Theta_l}\|\\ &\leq C_1\prod_{l\in u_j}\|e^{j_l\Theta_l}\|,\end{aligned} \tag{15}$$

since when $l \notin u_j$, it holds that $0 \leq j_l \leq t_l$. Moreover, also the number of possible index sets u_j is finite, so we may simply define C_1 in terms of maximums above. Denote the smallest eigenvalue of a positive definite matrix A by $\lambda_1(A)$. In addition, let $\lambda_1 = \min_l\{\lambda_1(\Theta_l)\}$. Since $j_l < 0$ for $l \in u_j$, this gives

$$\begin{aligned}\prod_{l\in u_j}\|e^{j_l\Theta_l}\| &= \prod_{l\in u_j} e^{j_l\lambda_1(\Theta_l)} = e^{\sum_{l\in u_j} j_l\lambda_1(\Theta_l)}\\ &\leq e^{\lambda_1\sum_{l\in u_j} j_l} \leq e^{\lambda_1\sum_{l=1}^{N} j_l} = e^{\lambda_1\left(\sum_{l=1}^{N} t_l - k\right)}.\end{aligned} \tag{16}$$

Combination of (14), (15), and (16) gives

$$\mathbb{P}\left(\sup_{\substack{\sum j=\sum t-k\\ j\leq t}} \|e^{j*\Theta}\Delta_j G\| > \epsilon\right) \leq \sum_{\substack{\sum j=\sum t-k\\ j\leq t}} \mathbb{P}\left(\|\Delta_{\mathbf{1}} G\| > \frac{\epsilon}{C_1 e^{\lambda_1(\sum t-k)}}\right), \tag{17}$$

where the summands are now independent of the summation index and consequently, we next evaluate the number of multi-indices j with $j \leq t$ and

$\sum j = \sum t - k$. Set $\tilde{j} = j - t$. Then equivalently, we want to count $\tilde{j}$ such that $\tilde{j} \leq 0$ and $\sum \tilde{j} = \sum(j-t) = -k$. This corresponds to finding the number of *weak compositions of k into N parts*, and that number is

$$\binom{k+N-1}{k} = \frac{(k+N-1)!}{(N-1)!k!} = \mathcal{O}(k^{N-1}).$$

When k is large enough, this together with (17) yields

$$\begin{aligned}
&\mathbb{P}\left(\sup_{\substack{\sum j=\sum t-k\\ j\leq t}} \|e^{j*\Theta}\Delta_j G\| > \epsilon\right) \leq C_2 k^{N-1}\mathbb{P}\left(\|\Delta_{\mathbf{1}}G\| > \frac{\epsilon}{C_1 e^{\lambda_1(\sum t-k)}}\right)\\
&= C_2 k^{N-1}\mathbb{P}\left(\|\Delta_{\mathbf{1}}G\|\mathbb{1}_{\{\|\Delta_{\mathbf{1}}G\|\neq 0\}} > \frac{\epsilon}{C_1 e^{\lambda_1(\sum t-k)}}\right)\\
&= C_2 k^{N-1}\mathbb{P}\left(\ln\|\Delta_{\mathbf{1}}G\|\mathbb{1}_{\{\|\Delta_{\mathbf{1}}G\|\neq 0\}} > \ln\epsilon - \ln C_1 - \lambda_1\left(\sum t - k\right)\right),
\end{aligned}$$

where, when k is large enough,

$$\ln\epsilon - \ln C_1 - \lambda_1\sum t + \lambda_1 k = k\left(\frac{\ln\epsilon - \ln C_1 - \lambda_1\sum t}{k} + \lambda_1\right) \geq C_3 k$$

for some $C_3 > 0$. Hence, for k large enough,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{\substack{\sum j=\sum t-k\\ j\leq t}} \|e^{j*\Theta}\Delta_j G\| > \epsilon\right) \leq C_2 k^{N-1}\mathbb{P}\left(\ln\|\Delta_{\mathbf{1}}G\|\mathbb{1}_{\{\|\Delta_{\mathbf{1}}G\|\neq 0\}} > C_3 k\right)\\
&= C_2 k^{N-1}\mathbb{P}\left(\ln\|\Delta_{\mathbf{1}}G\|\mathbb{1}_{\{\|\Delta_{\mathbf{1}}G\|>1\}} > C_3 k\right)\\
&\leq C_2 k^{N-1}\frac{\mathbb{E}\left(\ln\|\Delta_{\mathbf{1}}G\|\mathbb{1}_{\{\|\Delta_{\mathbf{1}}G\|>1\}}\right)^{N+\delta}}{(C_3 k)^{N+\delta}} = \frac{C}{k^{1+\delta}}
\end{aligned}$$

by Markov's inequality and (7). Then, Borel-Cantelli lemma gives

$$\sup_{\substack{\sum j=\sum t-k\\ j\leq t}} \|e^{j*\Theta}\Delta_j G\| \to 0$$

almost surely as $k \to \infty$, and furthermore

$$\sup_{k\geq 0}\sup_{\substack{\sum j=\sum t-k\\ j\leq t}} \|e^{j*\Theta}\Delta_j G\| = \sup_{j\leq t}\|e^{j*\Theta}\Delta_j G\| < \infty$$

almost surely, where $\Theta \in (S_+^n)_c^N$ is arbitrary.

Now we are in a position to show that (5) converges almost surely. We have that

$$\sum_{j=-M}^{t}\|e^{j*\Theta}\Delta_j G\| \le \sum_{j=-M}^{t}\|e^{\frac{1}{2}j*\Theta}\Delta_j G\|\|e^{\frac{1}{2}j*\Theta}\| \le \sup_{j\le t}\|e^{\frac{1}{2}j*\Theta}\Delta_j G\|\sum_{j=-M}^{t}\|e^{\frac{1}{2}j*\Theta}\| \le C\sum_{j=-M}^{t}\|e^{\frac{1}{2}j*\Theta}\| \tag{18}$$

almost surely. We write the above sum in 2^N parts as

$$\sum_{j=-M}^{t}\|e^{\frac{1}{2}j*\Theta}\| = \sum_{u\subseteq\{1,\dots,N\}}\sum_{j_u=-M_u}^{-\mathbf{1}}\sum_{j_{-u}=\mathbf{0}}^{t_{-u}}\|e^{\frac{1}{2}j*\Theta}\|, \tag{19}$$

where j_l runs from $-M_l$ to -1 for $l\in u$ and from 0 to t_l for $l\notin u$. When $t_l<0$, the interpretation here is that $\sum_{j_l=0}^{t_l} = -\sum_{j_l=t_l}^{0}$. Now

$$\begin{aligned}\|e^{\frac{1}{2}j*\Theta}\| &= \|e^{\sum_{l\in u}\frac{1}{2}j_l\Theta_l+\sum_{l\notin u}\frac{1}{2}j_l\Theta_l}\| \le \|e^{\sum_{l\in u}\frac{1}{2}j_l\Theta_l}\|\|e^{\sum_{l\notin u}\frac{1}{2}j_l\Theta_l}\| \\ &\le C\|e^{\sum_{l\in u}\frac{1}{2}j_l\Theta_l}\| \le C\prod_{l\in u}\|e^{\frac{1}{2}j_l\Theta_l}\| = C\prod_{l\in u}e^{\frac{1}{2}j_l\lambda_1(\Theta_l)}\end{aligned}$$

similarly as in the earlier part of the proof. By combining with (18) and (19),

$$\sum_{j=-M}^{t}\|e^{j*\Theta}\Delta_j G\| \le C\sum_{u\subseteq\{1,\dots,N\}}\sum_{j_{-u}=\mathbf{0}}^{t_{-u}}\sum_{j_u=-M_u}^{-\mathbf{1}}\prod_{l\in u}e^{\frac{1}{2}j_l\lambda_1(\Theta_l)}$$

almost surely. As $M\to\infty$,

$$\lim_{M\to\infty}\sum_{j=-M}^{t}\|e^{j*\Theta}\Delta_j G\| \le C\sum_{u\subseteq\{1,\dots,N\}}\sum_{j_{-u}=\mathbf{0}}^{t_{-u}}C_u,$$

where the remaining sums are over finite sets showing that the series is absolutely convergent almost surely and thus, completing the proof. □

Proof of Corollary 2.19. Let $M>1$ be such that $(\ln x)^{N+\delta}<x$, when $x\ge M$. Then

$$\begin{aligned}&\mathbb{E}\left(\ln\|\Delta_{\mathbf{1}}G\|\mathbb{1}_{\{\|\Delta_{\mathbf{1}}G\|>1\}}\right)^{N+\delta} \\ &= \mathbb{E}\left(\ln\|\Delta_{\mathbf{1}}G\|\mathbb{1}_{\{\|\Delta_{\mathbf{1}}G\|\ge M\}}\right)^{N+\delta} + \mathbb{E}\left(\ln\|\Delta_{\mathbf{1}}G\|\mathbb{1}_{\{1<\|\Delta_{\mathbf{1}}G\|<M\}}\right)^{N+\delta} \\ &\le \mathbb{E}\|\Delta_{\mathbf{1}}G\| + (\ln M)^{N+\delta}\end{aligned}$$

□

Proof of Lemma 2.30. The unit cube increments of Y are of the form

$$\begin{aligned}\Delta_t Y &= \sum_{i\in\{0,1\}^N} (-1)^{\sum i} \sum_{j=-\infty}^{t-i} e^{j*\Theta}\Delta_j G \\ &= \sum_{i\in\{0,1\}^N} (-1)^{\sum i} \sum_{j_1=-\infty}^{t_1-i_1} \cdots \sum_{j_N=-\infty}^{t_N-i_N} e^{j*\Theta}\Delta_j G.\end{aligned} \tag{20}$$

Let us consider $e^{j*\Theta}\Delta_j G$, where $j_l = t_l$ for all $l \in M_+ \subset \{1,\dots,N\}$ with $|M_+| = m \neq N$. The term belongs to the N-fold sum of (20) for every i such that $i_l = 0$ for all $l \in M_+$. That is, m of the indices i_l are zeros and remaining $N-m$ indices can be zeros or ones. Taking into account of the alternating sign in (20), the total number of terms $e^{j*\Theta}\Delta_j G$ in (20) is

$$\binom{N-m}{0} - \binom{N-m}{1} + \cdots + \binom{N-m}{N-m} = 0, \quad N-m \text{ is even}$$

and

$$\binom{N-m}{0} - \binom{N-m}{1} + \cdots - \binom{N-m}{N-m} = 0, \quad N-m \text{ is odd}$$

by Lemma 4.1. If $m = N$, then the corresponding term $e^{t*\Theta}\Delta_t G$ belongs only to the N-fold sum with $i = \mathbf{0}$ completing the first part of the proof.

For the second part, we notice that

$$\begin{aligned}\sum_{j_1=-M_1}^{t_1} \cdots \sum_{j_N=-M_N}^{t_N} e^{j*\Theta}\Delta_j G &= \sum_{j_1=0}^{t_1+M_1} \cdots \sum_{j_N=0}^{t_N+M_N} e^{(t-j)*\Theta}\Delta_{t-j} G \\ &= e^{t*\Theta} \sum_{j_1=0}^{t_1+M_1} \cdots \sum_{j_N=0}^{t_N+M_N} e^{-j*\Theta}\Delta_{t-j} G.\end{aligned}$$

Furthermore

$$\begin{aligned}& e^{(t+s)*\Theta} \sum_{j_1=0}^{t_1+s_1+M_1} \cdots \sum_{j_N=0}^{t_N+s_N+M_N} e^{-j*\Theta}\Delta_{t+s-j} G \\ & \overset{\text{law}}{=} e^{s*\Theta} e^{t*\Theta} \sum_{j_1=0}^{t_1+s_1+M_1} \cdots \sum_{j_N=0}^{t_N+s_N+M_N} e^{-j*\Theta}\Delta_{t-j} G.\end{aligned}$$

Since $G \in \mathcal{G}_\Theta$, this yields

$$\begin{aligned} Y_{e^{t+s}} &= \lim_{M_1\to\infty} \dots \lim_{M_N\to\infty} e^{(t+s)*\Theta} \sum_{j_1=0}^{t_1+s_1+M_1} \cdots \sum_{j_N=0}^{t_N+s_N+M_N} e^{-j*\Theta} \Delta_{t+s-j} G \\ &\overset{\text{law}}{=} \lim_{M_1\to\infty} \dots \lim_{M_N\to\infty} e^{s*\Theta} e^{t*\Theta} \sum_{j_1=0}^{t_1+s_1+M_1} \cdots \sum_{j_N=0}^{t_N+s_N+M_N} e^{-j*\Theta} \Delta_{t-j} G \\ &= e^{s*\Theta} \lim_{\tilde{M}_1\to\infty} \dots \lim_{\tilde{M}_N\to\infty} e^{t*\Theta} \sum_{j_1=0}^{t_1+\tilde{M}_1} \cdots \sum_{j_N=0}^{t_N+\tilde{M}_N} e^{-j*\Theta} \Delta_{t-j} G = e^{s*\Theta} Y_{e^t}, \end{aligned}$$

where $\tilde{M} = M + s$. Treating multidimensional distributions similarly completes the proof. □

Proof of Lemma 2.32. First we show that the increments of G are stationary under the assumption $\Theta \in (S_+^n)_c^N$. Let $s \in \mathbb{Z}^N$. Then

$$\Delta_{t+s} G = e^{-(t+s)*\Theta} \Delta_{t+s} Y \overset{\text{law}}{=} e^{-(t+s)*\Theta} e^{s*\Theta} \Delta_t Y = e^{-t*\Theta} \Delta_t Y = \Delta_t G,$$

where we have used Θ-self-similarity of Y and the fact that the involved matrix exponents commute. Multidimensional distributions of ΔG can be treated similarly.

Next, we consider the convergence of the N-fold sum. We have that

$$\sum_{j_1=-M_1+1}^{t_1} \cdots \sum_{j_N=-M_N+1}^{t_N} e^{j*\Theta} \Delta_j G = \sum_{j_1=-M_1+1}^{t_1} \cdots \sum_{j_N=-M_N+1}^{t_N} \Delta_t Y = \Delta_{-M,t} Y \tag{21}$$

by Lemma 2.13. Furthermore

$$\Delta_{-M,t} Y = \sum_{i\in\{0,1\}^N} (-1)^{\sum i} Y_{e^{t-i(t+M)}}, \tag{22}$$

where

$$Y_{e^{t-i(t+M)}} \overset{\text{law}}{=} e^{-i(t+M)*\Theta} Y_{e^t} \tag{23}$$

and

$$\|e^{-i(t+M)*\Theta} Y_{e^t}\| \le \|e^{-i(t+M)*\Theta}\| \|Y_{e^t}\| = \|e^{-\sum_{l=1}^N i_l(t_l+M_l)\Theta_l}\| \|Y_{e^t}\|. \tag{24}$$

The matrices $i_l(t_l + M_l)\Theta_l$ are positive definite when $i_l = 1$ and M_l is large enough. Let $\lambda_1(A)$ denote the smallest eigenvalue of matrix A. As a direct consequence of Courant-Fischer min-max theorem

$$\lambda_1(A + B) \ge \max\{\lambda_1(A), \lambda_1(B)\}$$

***for any positive semidefinite matrices A and B. Hence, we get

$$\lambda_1\left(\sum_{l=1}^{N} i_l(t_l+M_l)\Theta_l\right) \geq \max_l\{\lambda_1(i_l(t_l+M_l)\Theta_l)\}.$$

Let $\sum_{l=1}^{N} i_l(t_l+M_l)\Theta_l = Q\Lambda Q^T$ be an eigendecomposition with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Then

$$\begin{aligned}\|e^{-\sum_{l=1}^{N} i_l(t_l+M_l)\Theta_l}\| &= \|Qe^{-\Lambda}Q^T\| = \|\mathrm{diag}(e^{-\lambda_i})\| = \max_i\{e^{-\lambda_i}\} = e^{-\lambda_1}\\ &\leq e^{-\max_l\{\lambda_1(i_l(t_l+M_l)\Theta_l)\}} = e^{-\max_l\{i_l(t_l+M_l)\lambda_1(\Theta_l)\}} \to 0,\end{aligned} \tag{25}$$

whenever $i \neq \mathbf{0}$ and $M \to \infty$. Now, from (23), (24), and (25),

$$\mathbb{P}\left(\|Y_{e^{t-i(t+M)}}\| \geq \epsilon\right) = \mathbb{P}\left(\|e^{-i(t+M)*\Theta}Y_{e^t}\| \geq \epsilon\right) \to 0,$$

whenever $i \neq \mathbf{0}$ and $M \to \infty$. By combining this with (21) and (22), we conclude that

$$\sum_{j=-M+\mathbf{1}}^{t} e^{j*\Theta}\Delta_j G = \sum_{i\in\{0,1\}^N} (-1)^{\sum i} Y_{e^{t-i(t+M)}} \to Y_{e^t}$$

in probability as $M \to \infty$. □

Proof of Lemma 2.34. The property that $G_t = 0$ on the hyperplanes with $t_l = 0$ for some l follows from the lower bound $j_u = 1$ of the summation in (11) resulting in an empty sum. Let $t \in \mathbb{Z}^N$ and consider a term $e^{-j*\Theta}\Delta_j Y$. We notice that it belongs to (11) if and only if $1 \leq j_l \leq t_l$ for all $l \in u$ and $t_l + 1 \leq j_l \leq 0$ for all $l \in -u$. In what follows, we write $\{t-i \geq 0\} = \{l : t_l - i_l \geq 0\}$. Now

$$\Delta_t G = \sum_{i\in\{0,1\}^N} (-1)^{\sum i}(-1)^{|\{t-i<0\}|} \sum_{j_{\{t-i\geq 0\}}=\mathbf{1}}^{(t-i)_{\{t-i\geq 0\}}} \sum_{j_{\{t-i<0\}}=(t-i)_{\{t-i<0\}}+\mathbf{1}}^{\mathbf{0}} e^{-j*\Theta}\Delta_j Y. \tag{26}$$

By the previous observation, (26) consists of terms $e^{-j*\Theta}\Delta_j Y$ for which $1 \leq j_l \leq t_l$ for all l such that $t_l \geq 1$, $j_l = 0$ for all l such that $t_l = 0$ and $t_l \leq j_l \leq 0$ for all l such that $t_l \leq -1$. Let $e^{-j*\Theta}\Delta_j Y$ be such term. Moreover, let M_+, M_0, and M_- be the set of indices l for which $t_l \geq 1, t_l = 0$, and $t_l \leq -1$, respectively. If $e^{-j*\Theta}\Delta_j Y$ belongs to the N-fold sum in (26) for $i \in \{0,1\}^N$, then the following observations hold true.

(i) It holds that $i_{M_0} = 1$. That is, $|M_0|$ of the indices i_l are ones. The corresponding contribution to the sign in (26) is $(-1)^{|M_0|}(-1)^{|M_0|} = 1$.
(ii) Let m_+ be the set of indices for which $j_l = t_l$ with $t_l \geq 1$. Now, it holds that $i_{m_+} = 0$. That is, $|m_+|$ of the indices are zeros having no contribution to the sign in (26). The indices $M_+ - m_+$ can be either zeros or ones.

(iii) Let m_- be the set of indices for which $j_l = t_l$ with $t_l \leq -1$. Now, it holds that $i_{m_-} = 1$. That is, $|m_-|$ of the indices are ones. The contribution to the sign in (26) is $(-1)^{|m_-|}(-1)^{|M_-|} = (-1)^{|m_-|+|M_-|}$. The indices $M_- - m_-$ can be either zeros or ones.

Now the total number of terms $e^{-j*\Theta}\Delta_j Y$ in (26) is

$$(-1)^{|m_-|+|M_-|}\left[\binom{|M_+ - m_+| + |M_- - m_-|}{0} - \binom{|M_+ - m_+| + |M_- - m_-|}{1} + \cdots \pm \binom{|M_+ - m_+| + |M_- - m_-|}{|M_+ - m_+| + |M_- - m_-|}\right], \tag{27}$$

where the sign of the last binomial coefficient depends on whether $|M_+ - m_+| + |M_- - m_-|$ is even or odd. Regardless, by Lemma 4.1, the result is zero unless $|M_+ - m_+| + |M_- - m_-| = 0$. In that case $M_+ = m_+$ and $M_- = m_-$ meaning that $j_l = t_l$ for all l, i.e., $e^{-j*\Theta}\Delta_j Y = e^{-t*\Theta}\Delta_t Y$. Now (27) gives

$$(-1)^{|m_-|+|M_-|}\binom{0}{0} = (-1)^{2|M_-|} = 1.$$

□

Proof of Theorems 2.21. The first claim of the theorem is obtained by combining Lemmas 2.30, 2.32, and 2.34. Let us consider $(\mathcal{M}_\Theta^{-1} \circ \mathcal{M}_\Theta)(Y)$ for a Θ-self-similar Y. Then, by Lemmas 2.13 and 2.34

$$\begin{aligned}
\mathcal{M}_\Theta^{-1}(\mathcal{M}_\Theta(Y))_t &= \lim_{M_1\to\infty} \cdots \lim_{M_N\to\infty} \sum_{j=-M+\mathbf{1}}^{t} e^{j*\Theta}\Delta_j(\mathcal{M}_\Theta(Y)) \\
&= \lim_{M_1\to\infty} \cdots \lim_{M_N\to\infty} \sum_{j=-M+\mathbf{1}}^{t} \Delta_j Y \\
&= \lim_{M_1\to\infty} \cdots \lim_{M_N\to\infty} \Delta_{-M,t} Y = Y_{e^t}
\end{aligned}$$

similarly as in the proof of Lemma 2.32.

Next, we consider $(\mathcal{M}_\Theta \circ \mathcal{M}_\Theta^{-1})(G)$ for $G \in \mathcal{G}_{\Theta,0}$. By Lemma 2.30, $Y_{e^t} = (\mathcal{M}_\Theta^{-1} G)_{e^t}$ is Θ-self-similar with $\Delta_t Y = e^{t*\Theta}\Delta_t G$. On the other hand, $\tilde{G}_t = (\mathcal{M}_\Theta Y)_t$ is in $\mathcal{G}_{\Theta,0}$ with $\Delta_t \tilde{G} = e^{-t*\Theta}\Delta_t Y$. Since $e^{-t*\Theta}$ is the inverse of $e^{t*\Theta}$, we get

$$e^{t*\Theta}\Delta_t \tilde{G} = \Delta_t Y = e^{t*\Theta}\Delta_t G$$

and furthermore $\Delta_t \tilde{G} = \Delta_t G$ for all $t \in \mathbb{Z}^N$. It remains to show that this implies $\tilde{G}_t = G_t$ for all $t \in \mathbb{Z}^N$. For this, let t be such that $t_l \neq 0$ for all l. Then, by

Remark 2.11,

$$G_t = \Delta_{\mathbf{0},t} G = (-1)^m \Delta_{\tilde{\mathbf{0}},\tilde{t}} G,$$

where m is the number of elements for which $t_l < 0$ and $\tilde{\mathbf{0}}$ and $\tilde{t}$ are the vectors where such elements are swapped so that $\tilde{t} \geq \tilde{\mathbf{0}}$ element-wise. Therefore, by Lemma 2.13,

$$G_t = (-1)^m \Delta_{\tilde{\mathbf{0}},\tilde{t}} G = (-1)^m \sum_{j=\tilde{\mathbf{0}}+\mathbf{1}}^{\tilde{t}} \Delta_j G = (-1)^m \sum_{j=\tilde{\mathbf{0}}+\mathbf{1}}^{\tilde{t}} \Delta_j \tilde{G} = \tilde{G}_t.$$

□

Proof of Theorems 2.26. Set $Q_t^{(N)} = \Delta_t G$. Then, by Definition 2.24

$$\begin{aligned} X_t &= \hat{\Theta} \star \hat{X}_t + \Delta_t G = \sum_{\substack{i \in \{0,1\}^N \\ i \neq \mathbf{0}}} (-1)^{1+\sum i} e^{-i*\Theta} X_{t-i} + Q_t^{(N)} \\ &= \sum_{i \in \{0,1\}^{N-1}} (-1)^{\sum i} e^{-(i,1)*\Theta} X_{t-i,t_N-1} \\ &\quad + \sum_{\substack{i \in \{0,1\}^{N-1} \\ i \neq \mathbf{0}}} (-1)^{1+\sum i} e^{-(i,0)*\Theta} X_{t-i,t_N} + Q_t^{(N)}, \end{aligned}$$

where e.g. $(i, 1) = (i_1, \ldots, i_{N-1}, 1)$ and $X_{t-i,t_N-1} = X_{t_1-i_1,\ldots,t_{N-1}-i_{N-1},t_N-1}$. From above we get

$$\begin{aligned} X_t + \sum_{\substack{i \in \{0,1\}^{N-1} \\ i \neq \mathbf{0}}} (-1)^{\sum i} e^{-(i,0)*\Theta} X_{t-i,t_N} &= \sum_{i \in \{0,1\}^{N-1}} (-1)^{\sum i} e^{-(i,0)*\Theta} X_{t-i,t_N} \\ = \sum_{i \in \{0,1\}^{N-1}} (-1)^{\sum i} e^{-(i,1)*\Theta} X_{t-i,t_N-1} + Q_t^{(N)} \\ = e^{-\Theta_N} \sum_{i \in \{0,1\}^{N-1}} (-1)^{\sum i} e^{-(i,0)*\Theta} X_{t-i,t_N-1} + Q_t^{(N)}, \end{aligned}$$

since the involved matrices commute. By denoting

$$Y_t^{(N)}(t_N) = \sum_{i \in \{0,1\}^{N-1}} (-1)^{\sum i} e^{-(i,0)*\Theta} X_{t-i,t_N}$$

we obtain

$$Y_t^{(N)}(t_N) = e^{-\Theta_N} Y_t^{(N)}(t_N - 1) + Q_{t,t_N}^{(N)}.$$

By iterating the above equation, we get for every $n \in \mathbb{N}$ that

$$\begin{aligned} Y_t^{(N)}(t_N) &= e^{-(n+1)\Theta_N} Y_t^{(N)}(t_N - n - 1) + \sum_{j_N=0}^{n} e^{-j_N\Theta_N} Q_{t,t_N-j_N}^{(N)} \\ &= e^{-(n+1)\Theta_N} Y_t^{(N)}(t_N - n - 1) + \sum_{k_N=t_N-n}^{t_N} e^{(k_N-t_N)\Theta_N} Q_{t,k_N}^{(N)}, \end{aligned} \tag{28}$$

where

$$\begin{aligned} e^{-(n+1)\Theta_N} Y_t^{(N)}(t_N - n - 1) &= e^{-t_N\Theta_N} e^{(t_N-n-1)\Theta_N} Y_t^{(N)}(t_N - n - 1) \\ &= e^{-t_N\Theta_N} e^{m\Theta_N} Y_t^{(N)}(m) \end{aligned}$$

with $m = t_N - n - 1 \to -\infty$ as $n \to \infty$. By the condition (i) of Theorem 2.26,

$$e^{m\Theta_N} Y_t^{(N)}(m) = e^{m\Theta_N} \sum_{i\in\{0,1\}^{N-1}} (-1)^{\sum i} e^{-(i,0)*\Theta} X_{t-i,m} \to 0$$

in probability as $m \to -\infty$. Hence, as $n \to \infty$, (28) yields

$$Y_t^{(N)}(t_N) = e^{-t_N\Theta_N} \sum_{j_N=-\infty}^{t_N} e^{j_N\Theta_N} Q_{t,j_N}^{(N)}.$$

Thus, the induction assumption that for $k \in \mathbb{N}$,

$$\begin{aligned} &\sum_{i\in\{0,1\}^{N-k}} (-1)^{\sum i} e^{-(i,\mathbf{0})*\Theta} X_{t-i,t_{N-k+1},\dots,t_N} \\ &= e^{-\sum_{l=1}^{k} t_{N-l+1}\Theta_{N-l+1}} \sum_{j_{N-k+1}=-\infty}^{t_{N-k+1}} \cdots \sum_{j_N=-\infty}^{t_N} e^{\sum_{l=1}^{k} j_{N-l+1}\Theta_{N-l+1}} Q_{t,j_{N-k+1},\dots,j_N}^{(N)} \\ &=: Q_t^{(N-k)} \end{aligned} \tag{29}$$

holds true when $k = 1$. Note that above $(i, \mathbf{0}) = (i_1, \dots, i_{N-k}, 0, \dots, 0)$. This gives

$$\begin{aligned} &\sum_{i\in\{0,1\}^{N-k-1}} (-1)^{\sum i} e^{-(i,\mathbf{0})*\Theta} X_{t-i,t_{N-k},\dots,t_N} \\ &= e^{-\Theta_{N-k}} \sum_{i\in\{0,1\}^{N-k-1}} (-1)^{\sum i} e^{-(i,\mathbf{0})*\Theta} X_{t-i,t_{N-k}-1,t_{N-k+1},\dots,t_N} + Q_t^{(N-k)}. \end{aligned}$$

Thus, by setting

$$Y_t^{(N-k)}(t_{N-k}) = \sum_{i\in\{0,1\}^{N-k-1}} (-1)^{\sum i} e^{-(i,\mathbf{0})*\Theta} X_{t-i,t_{N-k},\dots,t_N}$$

we obtain that

$$Y_t^{(N-k)}(t_{N-k}) = e^{-\Theta_{N-k}} Y_t^{(N-k)}(t_{N-k} - 1) + Q_{t_1,\dots,t_{N-k},\dots,t_N}^{(N-k)}.$$

Again by iteration, it holds for every $n \in \mathbb{N}$ that

$$Y_t^{(N-k)}(t_{N-k}) = e^{-(n+1)\Theta_{N-k}} Y_t^{(N-k)}(t_{N-k} - n - 1) + \sum_{j_{N-k}=0}^{n} e^{-j_{N-k}\Theta_{N-k}} Q^{(N-k)}_{t_1,\dots,t_{N-k-1},t_{N-k}-j_{N-k},t_{N-k+1},\dots,t_N},$$

where similarly as before

$$e^{-(n+1)\Theta_{N-k}} Y_t^{(N-k)}(t_{N-k} - n - 1) \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty.$$

Hence

$$\begin{aligned} Y_t^{(N-k)}(t_{N-k}) &= \sum_{j_{N-k}=0}^{\infty} e^{-j_{N-k}\Theta_{N-k}} Q^{(N-k)}_{t_1,\dots,t_{N-k-1},t_{N-k}-j_{N-k},t_{N-k+1},\dots,t_N} \\ &= e^{-t_{N-k}\Theta_{N-k}} \sum_{j_{N-k}=-\infty}^{t_{N-k}} e^{j_{N-k}\Theta_{N-k}} Q^{(N-k)}_{t_1,\dots,t_{N-k-1},j_{N-k},t_{N-k+1},\dots,t_N}. \end{aligned}$$

By the definitions of $Y_t^{(N-k)}$ and $Q_t^{(N-k)}$ this gives directly that

$$\begin{aligned} &\sum_{i\in\{0,1\}^{N-k-1}} (-1)^{\sum i} e^{-(i,\mathbf{0})*\Theta} X_{t-i,t_{N-k},\dots,t_N} \\ &= e^{-\sum_{l=1}^{k+1} t_{N-l+1}\Theta_{N-l+1}} \sum_{j_{N-k}=-\infty}^{t_{N-k}} \cdots \sum_{j_N=-\infty}^{t_N} e^{\sum_{l=1}^{k+1} j_{N-l+1}\Theta_{N-l+1}} Q^{(N)}_{t,j_{N-k},\dots,j_N}, \end{aligned}$$

which completes the induction step. Now, by choosing $k = N$ in (29) we get

$$\begin{aligned} X_t &= e^{-\sum_{l=1}^{N} t_{N-l+1}\Theta_{N-l+1}} \sum_{j=-\infty}^{t} e^{\sum_{l=1}^{N} j_{N-l+1}\Theta_{N-l+1}} Q_j^{(N)} \\ &= e^{-t*\Theta} \sum_{j=-\infty}^{t} e^{j*\Theta} \Delta_j G = (\mathcal{L}_\Theta^{-1}(\mathcal{M}_\Theta^{-1} G))_t \end{aligned}$$

and thus, X is stationary. □

We utilize the present value and the corresponding increment in order to define the notion of "previous value" of a field.

Definition 4.2. *Let $X = (X_t)_{t\in\mathbb{Z}^N}$ and $t \in \mathbb{Z}^N$. We call the random vector*

$$X_t^- := X_t - \Delta_t X = \sum_{\substack{(i_1,\dots,i_N)\in\{0,1\}^N \\ (i_1,\dots,i_N)\neq \mathbf{0}}} (-1)^{1+\sum_{l=1}^{N} i_l} X_{t_1-i_1,\dots,t_N-i_N}.$$

as the "previous value" of X at t.

Proof of Theorems 2.28. Let $Y = \mathcal{L}_\Theta X$. Then

$$\begin{aligned}\Delta_t X = X_t - X_t^- &= e^{-t*\Theta} Y_{e^t} - X_t^- - e^{-t*\Theta}\Delta_t Y + e^{-t*\Theta}\Delta_t Y \\ &= e^{-t*\Theta}(Y_{e^t} - \Delta_t Y) - X_t^- + e^{-t*\Theta}\Delta_t Y\end{aligned}$$

giving

$$\Delta_t X + X_t^- = X_t = e^{-t*\Theta}(Y_{e^t} - \Delta_t Y) + e^{-t*\Theta}\Delta_t Y. \tag{30}$$

Define $G = \mathcal{M}_\Theta(Y) = (\mathcal{M}_\Theta \circ \mathcal{L}_\Theta)(X)$. Then $\Delta_t G = e^{-t*\Theta}\Delta_t Y$. In addition, since $\Theta \in (S_+^n)_c^N$,

$$\begin{aligned}e^{-t*\Theta}(Y_{e^t} - \Delta_t Y) &= e^{-t*\Theta} \sum_{\substack{i\in\{0,1\}^N \\ i\neq \mathbf{0}}} (-1)^{1+\sum i} Y_{e^{t-i}} \\ &= \sum_{\substack{i\in\{0,1\}^N \\ i\neq \mathbf{0}}} (-1)^{1+\sum i} e^{-i*\Theta} e^{-(t-i)*\Theta} Y_{e^{t-i}} \\ &= \sum_{\substack{i\in\{0,1\}^N \\ i\neq \mathbf{0}}} (-1)^{1+\sum i} e^{-i*\Theta} X_{t-i} = \hat{\Theta} \star \hat{X}_t.\end{aligned}$$

From (30) we conclude that

$$X_t = \hat{\Theta} \star \hat{X}_t + \Delta_t G$$

with $G = (\mathcal{M}_\Theta \circ \mathcal{L}_\Theta)(X)$. □

Proof of Theorems 2.29. By Theorems 2.26 and 2.28, it remains to prove uniqueness of G in 2.28. Let X be stationary and $G, \tilde{G} \in \mathcal{G}_{\Theta,0}$ satisfy (9). Then, by Theorem 2.26,

$$(\mathcal{L}_\Theta^{-1} \circ \mathcal{M}_\Theta^{-1})(G) = X = (\mathcal{L}_\Theta^{-1} \circ \mathcal{M}_\Theta^{-1})(\tilde{G})$$

giving

$$G = (\mathcal{M}_\Theta \circ \mathcal{L}_\Theta)(X) = \tilde{G}.$$

□

Disclosure statement

The authors report there are no competing interests to declare.

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