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AUTHOR VORRAPAN CHANDEE, XIANNAN LI, KAISA MATOMÄKI,
AND MAKSYM RADZIWIŁŁ

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THE EIGHTH MOMENT OF DIRICHLET L -FUNCTIONS II

VORRAPAN CHANDEE, XIANNAN LI, KAISA MATOMÄKI,
AND MAKSYM RADZIWIŁŁ

ABSTRACT. We prove an asymptotic formula for the eighth moment of Dirichlet L -functions averaged over primitive characters χ modulo q , over all moduli $q \leq Q$ and with a short average on the critical line. Previously the same result was shown conditionally on the Generalized Riemann Hypothesis by the first two authors.

1. INTRODUCTION

Moments of L -functions have attracted a great deal of attention. Not only do they have numerous applications, but they have also their own intrinsic interest. The first moments studied were naturally those of the Riemann zeta function, which are averages of the form

$$I_k(T) := \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt.$$

An asymptotic formula for $I_k(T)$ was proven for $k = 1$ by Hardy and Littlewood and for $k = 2$ by Ingham (see e.g. [12, Chapter VII]). Despite considerable effort, such an asymptotic formula is still not known for any other value of k .

The situation for other L -functions is very similar; asymptotics are only available for small values of k , and often only when averaged over a suitable family. In case of Dirichlet L -functions, Conrey, Iwaniec and Soundararajan [4] have proven an asymptotic formula for the sixth moment with an averaging over characters $\chi \pmod{q}$, over all moduli $q \leq Q$ and with a short average on the critical line. They stated in the paper that "A challenging problem is to obtain a similar asymptotic formula...for the eighth moment." Later, the first two authors [1] have proved an asymptotic formula for the eighth moment, conditionally on the Generalized Riemann Hypothesis (GRH). The aim of this paper is to provide an unconditional proof of this eighth moment result.

Before stating our result, let us introduce some notation. Let $\chi \pmod{q}$ be a primitive even¹ Dirichlet character, and let (for $\operatorname{Re} s > 1$),

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

be the Dirichlet L -function associated to it. Then the completed L -function

$$\Lambda\left(\frac{1}{2} + s, \chi\right) := \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{1}{4} + \frac{s}{2}\right) L\left(\frac{1}{2} + s, \chi\right)$$

satisfies the functional equation

$$(1) \quad \Lambda\left(\frac{1}{2} + s, \chi\right) = \varepsilon_\chi \Lambda\left(\frac{1}{2} - s, \bar{\chi}\right),$$

where $|\varepsilon_\chi| = 1$.

Let $\sum_{\chi \pmod{q}}^b$ denote a sum over primitive even Dirichlet characters with modulus q , and $\phi^b(q)$ denote the number of primitive even Dirichlet characters with modulus q . From [3], one may derive the conjecture that as $q \rightarrow \infty$ with $q \not\equiv 2 \pmod{4}$,

$$\frac{1}{\phi^b(q)} \sum_{\chi \pmod{q}}^b |L\left(\frac{1}{2}, \chi\right)|^8 \sim 24024 a_4 \prod_{p|q} \frac{\left(1 - \frac{1}{p}\right)^7}{\left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right)} \frac{(\log q)^{16}}{16!},$$

where

$$a_4 := \prod_p \left(1 - \frac{1}{p}\right)^9 \left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right).$$

Toward this conjecture we prove the following asymptotic formula when there is an additional q -average as well as a t -average which is very short thanks to the rapid decay of the Γ -function. Exactly the same theorem was shown in [1] conditionally on the Generalized Riemann Hypothesis.

Theorem 1.1. *Let $\varepsilon > 0$ and let Ψ be a smooth function compactly supported in $[1, 2]$. Then*

$$(2) \quad \begin{aligned} & \sum_q \Psi\left(\frac{q}{Q}\right) \sum_{\chi \pmod{q}}^b \int_{-\infty}^{\infty} |\Lambda\left(\frac{1}{2} + it, \chi\right)|^8 dt \\ &= 24024 a_4 \sum_q \Psi\left(\frac{q}{Q}\right) \prod_{p|q} \frac{\left(1 - \frac{1}{p}\right)^7}{\left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right)} \phi^b(q) \frac{(\log q)^{16}}{16!} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \right|^8 dt \\ & \quad + O_\varepsilon(Q^2 (\log Q)^{15+\varepsilon}). \end{aligned}$$

¹The restriction to even characters is only for convenience so that the completed L -function has the same shape for all characters involved — odd characters could be treated in exactly the same way.

Remark 1.2. Note that the main term in the theorem is of the order $Q^2(\log Q)^{16}$ and one obtains as a corollary that

$$\begin{aligned} & \sum_{q \leq Q} \sum_{\chi \pmod{q}}^b \int_{-\infty}^{\infty} |\Lambda(\tfrac{1}{2} + it, \chi)|^8 dt \\ & \sim 24024a_4 \sum_{q \leq Q} \prod_{p|q} \frac{\left(1 - \frac{1}{p}\right)^7}{\left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right)} \phi^b(q) \frac{(\log q)^{16}}{16!} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \right|^8 dt. \end{aligned}$$

In the proof of Theorem 1.1, we start by truncating the Dirichlet series arising from the approximate functional equation. This truncation introduces an error term of size around $Q^2(\log Q)^{15+\epsilon}$. It thus remains an open problem to obtain the lower order terms in the asymptotic; in particular, we anticipate that obtaining a power saving error term would be a challenging task.

Remark 1.3. In [2], the present authors will remove the t -average from the sixth moment in the work of Conrey, Iwaniec, and Soundararajan [4]. In particular, we will show that

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}}^b |L(\tfrac{1}{2}, \chi)|^6 \sim 42a_3 \sum_{q \leq Q} \prod_{p|q} \frac{\left(1 - \frac{1}{p}\right)^5}{\left(1 + \frac{4}{p} + \frac{1}{p^2}\right)} \phi^b(q) \frac{(\log q)^9}{9!},$$

where

$$a_3 = \prod_p \left(1 - \frac{1}{p^4}\right) \left(1 + \frac{4}{p} + \frac{1}{p^2}\right).$$

However, it remains challenging to remove the t -average for the eighth moment.

2. A SKETCH OF THE PROOF

In this section we provide a sketch of the proof where we ignore various technicalities such as complicated smooth weights, the inclusion-exclusion within the orthogonality over primitive characters and a number of coprimality conditions and common divisors.

Roughly speaking, after applying the approximate functional equation, we need to understand sums of the form

$$(3) \quad Q \sum_q \Psi\left(\frac{q}{Q}\right) \sum_{\substack{m \leq Q^2 \\ m \equiv n \pmod{q}}} \sum_{\substack{n \leq Q^2 \\ n \equiv n \pmod{q}}} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}},$$

where $\tau_4(n) = \sum_{n=n_1 n_2 n_3 n_4} 1$. Here we were able to make the restriction $m, n \leq Q^2$ instead of just $mn \leq Q^4$ thanks to the t -average in the theorem.

The diagonal contribution $m = n$ in (3) is fairly easy to understand, and in this sketch we shall concentrate on the non-diagonal contribution. Let

$\varepsilon_0 > 0$ be small but fixed. Following [1] the sums over m and n in (3) can be truncated to

$$m, n \leq N := \frac{Q^2}{\exp((\log Q)^{\varepsilon_0})}$$

using the multiplicative large sieve. Still following [1] (and [4]) we apply in the most critical range the complementary divisor trick. That is we write $m - n = hq$ in (3) and replace the congruence condition modulo q with a congruence condition modulo h . Note that $h \ll N/q$ is smaller than Q , so we have a reduction in the arithmetic conductor.

After switching to the complementary divisor, we express the congruence condition modulo h using characters modulo h , so that roughly we want to study

$$(4) \quad Q \sum_{h \leq 2N/Q} \frac{1}{\phi(h)} \sum_{\chi \pmod{h}} \sum_{m, n} \frac{\tau_4(m)\tau_4(n)\chi(m)\overline{\chi(n)}}{\sqrt{mn}} \Psi\left(\frac{|m-n|}{hQ}\right) \Psi_1\left(\frac{m}{N}\right) \Psi_1\left(\frac{n}{N}\right),$$

where Ψ_1 is smooth and supported on $[0, 1]$.

The principal characters give a main term contribution. In this sketch we concentrate on the non-principal characters. The smooth factor

$$h(m, n) := \Psi\left(\frac{|m-n|}{hQ}\right) \Psi_1\left(\frac{m}{N}\right) \Psi_1\left(\frac{n}{N}\right)$$

restricts m and n to being within distance $2hQ$ from each other. Morally, the short interval type condition $|m-n| \ll 2hQ$ introduces an archimedean conductor of size $T = \frac{N}{hQ}$. This simplification is not obvious at first sight and requires an optimal treatment of the two-variable Mellin transform of $h(m, n)$ as in Lemma 6.3. This is one of the main components which allows us to remove the need for GRH.

The hybrid conductor is then $hT \asymp \frac{N}{Q}$, and this is still smaller than the original conductor Q . It is important that the sums of length $N \asymp \frac{Q^2}{\exp((\log Q)^{\varepsilon_0})}$ are long compared to the hybrid conductor. In particular, applying Fourier analysis to such a sum with $\tau_4(n)$ as the coefficient produces dual sums of length $\frac{(N/Q)^4}{N} \asymp \frac{N^3}{Q^4} \ll \frac{Q^2}{\exp(3(\log Q)^{\varepsilon_0})}$, and this is shorter than the length of the original sum $N = \frac{Q^2}{\exp((\log Q)^{\varepsilon_0})}$. Actually for technical convenience we use the approximate functional equation rather than the functional equation, and this gives us sums of length $\left(\frac{N}{Q}\right)^2 \asymp \frac{Q^2}{\exp(2(\log Q)^{\varepsilon_0})}$, which still suffices. This is the main motivation for the more technical arguments that follow.

To be more precise, we will reduce our problem to that of bounding a mean square of a corresponding Dirichlet series, in the spirit of arithmetic problems on almost all short intervals (see e.g. [5, Lemma 9.3]). Indeed, one

can show that the Mellin transform

$$(5) \quad \tilde{h}(s_1, s_2) = \int_0^\infty \int_0^\infty h(x, y) x^{s_1} y^{s_2} \frac{dy}{y} \frac{dx}{x}$$

converges for $\operatorname{Re} s_i > 0$ and satisfies, for $\operatorname{Re} s_i \in (0, 100)$,

$$(6) \quad \tilde{h}(s_1, s_2) \ll \frac{1}{\operatorname{Re} s_1 \cdot \operatorname{Re} s_2} \left(\frac{N}{hQ} \right)^{k-1} \frac{1}{\max\{|s_1|, |s_2|\}^k |s_1 + s_2|^l} N^{\operatorname{Re} s_1 + \operatorname{Re} s_2}$$

for any integers $k \geq 1$ and $l \geq 0$. In showing this, one can assume $|s_1| \geq |s_2|$, and in this case (6) follows by applying in (5) first partial integration k times with respect to x , then substituting $y = xz$ and finally applying partial integration l times with respect to x . A similar argument with our more complicated weight function can be found from the proof of Lemma 6.3.

Now by the Mellin inversion, the non-principal characters contribute to (4)

$$Q \sum_{h \leq 2N/Q} \frac{1}{\phi(h)} \sum_{\substack{\chi \pmod{h} \\ \chi \neq \chi_0}} \int_{(1/2+\varepsilon)} \int_{(1/2+\varepsilon)} L(1/2+s_1, \chi)^4 L(1/2+s_2, \bar{\chi})^4 \tilde{h}(s_1, s_2) ds_1 ds_2.$$

Moving the contours close to $\operatorname{Re} s_i = 0$, plugging in the bound (6) for $\tilde{h}(s_1, s_2)$, and using the inequality $xy \leq |x|^2 + |y|^2$, this can be seen to be essentially

$$\ll Q \sum_{h \leq 2N/Q} \frac{1}{\phi(h)} \sum_{\chi \pmod{h}} \frac{hQ}{N} \int_{-N/(hQ)}^{N/(hQ)} |L(1/2 + it, \chi)|^8 dt.$$

Hence our task more-or-less reduces to showing that, for any $H \leq N/Q$ and $C \geq 1$, we have

$$(7) \quad \frac{Q^2}{N} \sum_{h \sim H} \sum_{\chi \pmod{h}} \int_{-T}^T |L(1/2 + it, \chi)|^8 dt \ll \frac{Q^2}{(\log Q)^C} \quad \text{for } T = N/(HQ).$$

Here and later the notation $h \sim H$ in summations means that $H < h \leq 2H$. Since $HT = N/Q$, the approximate functional equation morally allows us to approximate $|L(1/2 + it, \chi)|^4$ by $|\sum_{n \ll (N/Q)^2} \chi(n) \tau_4(n) n^{-1/2-it}|$ (see Proposition 3.2 below for a rigorous argument) and hence by the large sieve (Lemma 3.1 below) the left hand side of (7) can be shown to be

$$\ll \frac{Q^2}{N} \left(H^2 T + \left(\frac{N}{Q} \right)^2 \right) (\log N)^{O(1)} \ll (QH + N) (\log N)^{O(1)}.$$

We recall that $H \leq N/Q$ and $N = \frac{Q^2}{\exp((\log Q)^{\varepsilon_0})}$, so the above is $O(Q^2/(\log Q)^C)$ for any $C \geq 1$.

The current paper has a lot in common with [1] and we freely borrow results from there, so the reader may want to have [1] at hand. However, if the reader is ready to take those results granted or to work them out, the current paper can be read alone.

Throughout the paper, ε denotes a small positive real number. Furthermore ε_0 and Δ_0 will be fixed positive constants that are chosen later.

3. LARGE SIEVE AND UPPER BOUNDS FOR MOMENTS

Let us first recall the hybrid large sieve (see e.g. [9, Theorem 5.1]).

Lemma 3.1. *Let $T, Q \geq 1$. For any complex coefficients a_n with $\sum_{n=1}^{\infty} |a_n| < \infty$, one has*

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}}^* \frac{q}{\phi(q)} \int_{-T}^T \left| \sum_{n=1}^{\infty} a_n \chi(n) n^{it} \right|^2 \ll \sum_{n=1}^{\infty} (Q^2 T + n) |a_n|^2.$$

In particular, for any $N \geq 1$ and any complex coefficients a_n , one has

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}}^* \frac{q}{\phi(q)} \int_{-T}^T \left| \sum_{n \leq N} a_n \chi(n) n^{it} \right|^2 \ll (Q^2 T + N) \sum_{n \leq N} |a_n|^2.$$

The following proposition gives an upper bound for the eight moment of Dirichlet L -functions averaged over χ, q and t . It will be used in bounding the off-diagonal terms in Section 6.3.

Proposition 3.2. *For $Q, T \geq 3$, one has*

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} \int_0^T \left| L\left(\frac{1}{2} + c + it, \chi\right) \right|^8 dt \ll Q^2 T^2 (\log(QT))^{16},$$

whenever $0 \leq c \leq 1/100$.

Remark 3.3. In [1] it was shown that conditionally on GRH, one has

$$(8) \quad \sum_{\chi \pmod{q}} \int_0^T \left| L\left(\frac{1}{2} + c + it, \chi\right) \right|^8 dt \ll qT (\log(qT))^{16+\varepsilon}.$$

The unconditional Proposition 3.2 suffices for us since we will afford to lose a factor T due to the decay in $|s_1 + s_2|$ coming from Lemma 6.3 below. On the other hand the fact that we need an additional average over q in Proposition 3.2 compared to (8) will cause us some minor technical difficulties (see in particular Remark 6.5).

Proof of Proposition 3.2. The contribution from $q = 1$ is acceptable by known moment bound for the Riemann zeta function (see Lemma 3.5 below). Moreover, the part with bounded t can be easily dealt with using the approximate functional equation and the large sieve (see [7, Theorem 7.34], the proof works just as well with 16 in place of 17).

Hence, by dyadic splitting, it suffices to prove, for $Q \geq 3/2$ and $T \geq 3$,

$$(9) \quad \sum_{q \sim Q} \sum_{\chi \pmod{q}} \int_T^{2T} \left| L\left(\frac{1}{2} + c + it, \chi\right) \right|^8 dt \ll Q^2 T^2 (\log(QT))^{16}.$$

As usual, we first reduce to primitive characters; we claim that (9) follows once we have shown that

$$(10) \quad \sum_{q \sim Q} \sum_{\chi \pmod{q}}^* \int_T^{2T} \left| L\left(\frac{1}{2} + c + it, \chi\right) \right|^8 dt \ll Q^2 T^2 (\log(QT))^{16}.$$

Indeed, the left hand side of (9) is

$$(11) \quad \leq \sum_{r \leq Q} \sum_{q_1 \sim \frac{Q}{r}} \sum_{\chi_1 \pmod{q_1}}^* \int_T^{2T} \left| L\left(\frac{1}{2} + c + it, \chi_1\right) \right|^8 \prod_{p|r} \left(1 + \frac{1}{p^{1/2}}\right)^8 dt$$

Applying (10), this is at most

$$Q^2 T^2 (\log(QT))^{16} \sum_{r \leq Q} \frac{1}{r^2} \prod_{p|r} \left(1 + \frac{1}{p^{1/2}}\right)^8 \ll Q^2 T^2 (\log(QT))^{16}$$

as claimed. Hence it suffices to prove (10). For χ a primitive character mod q , by the approximate functional equation we have morally that

$$\left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 \ll \left| \sum_{n \ll (TQ)^2} \frac{\tau_4(n) \chi(n)}{n^{1/2+it}} \right| + \left| \sum_{n \ll (TQ)^2} \frac{\tau_4(n) \bar{\chi}(n)}{n^{1/2-it}} \right|.$$

If this was true, the claim (10) (for $c = 0$) would follow from the hybrid large sieve (Lemma 3.1). However, there is a technical issue that the Dirichlet polynomials in the approximation depend mildly on t and q . To proceed rigorously, we use a method of Ramachandra [8].

Let us concentrate on the sum in (10) over even characters, the odd characters being handled similarly. By the functional equation (1), we have

$$L\left(\frac{1}{2} + s, \chi\right)^4 = F\left(\frac{1}{2} + s\right) L\left(\frac{1}{2} - s, \bar{\chi}\right)^4$$

with

$$F\left(\frac{1}{2} + s\right) := \varepsilon_\chi^4 \left(\frac{\pi}{q}\right)^{4s} \frac{\Gamma\left(\frac{1}{4} - \frac{s}{2}\right)^4}{\Gamma\left(\frac{1}{4} + \frac{s}{2}\right)^4},$$

where $|\varepsilon_\chi| = 1$. For further convenience, let us record here Stirling's formula which gives, for $|\text{Arg}(z)| < \pi - \varepsilon$,

$$(12) \quad \Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z (1 + O(1/z)) \ll \frac{|z|^{\text{Re } z - 1/2}}{e^{\text{Re } z}} \exp(-\text{Im } z \cdot \text{Arg}(z)).$$

In particular, for $\text{Re } s \in [-1, 1/3]$ and $|\text{Im } s| \geq 10$, we have

$$(13) \quad \frac{\Gamma\left(\frac{1}{4} - \frac{s}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{s}{2}\right)} \asymp \frac{\exp(\text{Re}(-\frac{1}{4} - \frac{s}{2}) \log|\frac{1}{4} - \frac{s}{2}| - \text{Im}(\frac{-s}{2}) \text{Arg}(\frac{1}{4} - \frac{s}{2}))}{\exp(\text{Re}(-\frac{1}{4} + \frac{s}{2}) \log|\frac{1}{4} + \frac{s}{2}| - \text{Im}(\frac{s}{2}) \text{Arg}(\frac{1}{4} + \frac{s}{2}))}$$

and

$$\log \left| \frac{1}{4} \pm \frac{s}{2} \right| = \log(|s|+1) + O(1) \quad \text{and} \quad \text{Arg} \left(\frac{1}{4} \pm \frac{s}{2} \right) = \text{Arg} \left(\pm \frac{s}{2} \right) + O \left(\frac{1}{|s|+1} \right).$$

Noticing that the left hand side of (13) stays bounded when $\operatorname{Re} s \in [-1, 1/3]$ and $|\operatorname{Im} s| < 10$, we obtain that, for any s with $\operatorname{Re} s \in [-1, 1/3]$,

$$(14) \quad \frac{\Gamma(\frac{1}{4} - \frac{s}{2})}{\Gamma(\frac{1}{4} + \frac{s}{2})} \ll \exp(-\operatorname{Re}(s) \log(|s| + 1)) \asymp (|s| + 1)^{-\operatorname{Re} s}.$$

The starting point in the proof of (10) is the following lemma.

Lemma 3.4. *Let $0 \leq c \leq 1/100$, let χ be an even primitive Dirichlet character of modulus $q \in \mathbb{N}$, and let $t \in \mathbb{R}$. Then for all $X > 0$,*

$$(15) \quad \begin{aligned} L(\tfrac{1}{2} + c + it, \chi)^4 &= \sum_{n=1}^{\infty} \frac{\tau_4(n)\chi(n)e^{-n/X}}{n^{1/2+c+it}} + F(\tfrac{1}{2} + c + it) \sum_{n \leq X} \frac{\tau_4(n)\bar{\chi}(n)}{n^{1/2-c-it}} \\ &\quad - \frac{1}{2\pi i} \int_{(-3/4)} F(\tfrac{1}{2} + c + it + w) \left(\sum_{n > X} \frac{\tau_4(n)\bar{\chi}(n)}{n^{1/2-w-c-it}} \right) \Gamma(w) X^w dw \\ &\quad - \frac{1}{2\pi i} \int_{(1/4)} F(\tfrac{1}{2} + c + it + w) \left(\sum_{n \leq X} \frac{\tau_4(n)\bar{\chi}(n)}{n^{1/2-c-w-it}} \right) \Gamma(w) X^w dw \\ &\quad - \mathbf{1}_{q=1} \operatorname{Res}_{w=1/2-c-it} \Gamma(w) X^w \zeta^4 \left(\frac{1}{2} + c + it + w \right), \end{aligned}$$

where $\mathbf{1}_{q=1}$ is 1 if $q = 1$, and 0 otherwise.

Proof. This follows similarly to [8, Theorem 2]: Notice first that by Mellin inversion (since $\Gamma(s)$ is the Mellin transform of the function e^{-x})

$$\sum_{n=1}^{\infty} \frac{\tau_4(n)\chi(n)e^{-n/X}}{n^{1/2+c+it}} = \frac{1}{2\pi i} \int_{(2)} L(\tfrac{1}{2} + c + it + w, \chi)^4 \Gamma(w) X^w dw.$$

We shift the integral to the line $\operatorname{Re} w = -\frac{3}{4}$, collecting a residue from the pole at $w = 0$ and in case $q = 1$ also from a pole at $w = 1/2 - c - it$. Applying also the functional equation to the integral on the line $\operatorname{Re} w = -\frac{3}{4}$, we see that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\tau_4(n)\chi(n)e^{-n/X}}{n^{1/2+c+it}} &= L(\tfrac{1}{2} + c + it, \chi)^4 + \mathbf{1}_{q=1} \operatorname{Res}_{w=1/2-c-it} \Gamma(w) X^w \zeta^4 \left(\frac{1}{2} + c + it + w \right) \\ &\quad + \frac{1}{2\pi i} \int_{(-3/4)} F(\tfrac{1}{2} + c + it + w, \chi) L(\tfrac{1}{2} - c - it - w, \bar{\chi})^4 \Gamma(w) X^w dw. \end{aligned}$$

Writing on the last line

$$L(\tfrac{1}{2} - c - it - w, \bar{\chi})^4 = \sum_{n \leq X} \frac{\tau_4(n)\bar{\chi}(n)}{n^{1/2-c-it-w}} + \sum_{n > X} \frac{\tau_4(n)\bar{\chi}(n)}{n^{1/2-c-it-w}},$$

the claim of the lemma reduces to the claim that

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(-3/4)} F\left(\frac{1}{2} + c + it + w, \chi\right) \sum_{n \leq X} \frac{\tau_4(n) \bar{\chi}(n)}{n^{1/2-c-it-w}} \Gamma(w) X^w dw \\ &= \frac{1}{2\pi i} \int_{(1/4)} F\left(\frac{1}{2} + c + it + w, \chi\right) \sum_{n \leq X} \frac{\tau_4(n) \bar{\chi}(n)}{n^{1/2-c-it-w}} \Gamma(w) X^w dw \\ & \quad - F\left(\frac{1}{2} + c + it, \chi\right) \sum_{n \leq X} \frac{\tau_4(n) \bar{\chi}(n)}{n^{1/2-c-it}}. \end{aligned}$$

But this is immediate from shifting the integration line, picking up a residue from a pole at $w = 0$. \square

Let us now return to (10) for even characters. Recall $Q \geq 3/2$ and $T \geq 3$. We take $X = (QT)^2$ and apply Lemma 3.4, writing (15) as $L(\frac{1}{2} + c + it, \chi)^4 = (J_1 + J_2 - J_3 - J_4)(c, t, \chi)$ (the fifth term in (15) always vanishes as $q \geq 2$, so we do not need to include it). Then it suffices to show that

$$\sum_{q \sim Q} \sum_{\chi \pmod{q}} \sum_{(\text{mod } q)}^b \int_T^{2T} |J_k(c, t, \chi)|^2 dt \ll (QT)^2 (\log(QT))^{16}$$

for $k = 1, 2, 3, 4$. First, by the large sieve (Lemma 3.1), we have

$$\begin{aligned} & \sum_{q \sim Q} \sum_{\chi \pmod{q}} \sum_{(\text{mod } q)}^b \int_T^{2T} |J_1(c, t, \chi)|^2 dt \ll \sum_{n=1}^{\infty} (Q^2 T + n) \frac{\tau_4(n)^2 e^{-2n/(Q^2 T^2)}}{n^{1+2c}} \\ & \ll Q^2 T^2 \sum_{n \leq Q^2 T^2} \frac{\tau_4(n)^2}{n} + \sum_{n > Q^2 T^2} \tau_4(n)^2 e^{-2n/(Q^2 T^2)} \ll Q^2 T^2 (\log(QT))^{16}. \end{aligned}$$

Furthermore, noting that, by (14), $|F(\frac{1}{2} + c + it)| \ll (q(1 + |t|))^{-4c}$, the large sieve (Lemma 3.1) implies that

$$\sum_{q \sim Q} \sum_{\chi \pmod{q}} \sum_{(\text{mod } q)}^b \int_T^{2T} |J_2(c, t, \chi)|^2 dt \ll (QT)^{2-8c} \sum_{n \leq Q^2 T^2} \frac{\tau_4(n)^2}{n^{1-2c}} \ll Q^2 T^2 (\log(QT))^{16}.$$

To deal with the remaining two cases we notice that, by (14) and (12), we have, for $\text{Re } w \in \{-3/4, 1/4\}$,

$$|F(\frac{1}{2} + it + c + w)| \ll q^{-4\text{Re } w - 4c} \cdot (1 + |t + \text{Im } w|)^{-4\text{Re } w - 4c} \quad \text{and} \quad |\Gamma(w)| \ll e^{-|\text{Im } w|}.$$

Splitting into cases according to whether $|\text{Im } w| \leq \frac{1}{2}|t|$ or not, we see that

$$|F(\frac{1}{2} + c + it + w) \Gamma(w) (QT)^{2w}| \ll (QT)^{-2\text{Re } w - 4c} e^{-\frac{1}{2}|\text{Im } w|}.$$

Hence by the Cauchy-Schwarz inequality, noting the rapid decay in $\text{Im } w$, and using the large sieve (Lemma 3.1) we have

$$\begin{aligned}
& \sum_{q \sim Q} \sum_{\chi \pmod{q}}^b \int_T^{2T} |J_3(c, t, \chi)|^2 dt \\
& \ll (QT)^{3-8c} \sum_{q \sim Q} \sum_{\chi \pmod{q}}^b \int_T^{2T} \left(\int_{-\infty}^{\infty} \left| \sum_{n > Q^2 T^2} \frac{\tau_4(n) \overline{\chi}(n)}{n^{5/4-c-it-iu}} \right| e^{-|u|/2} du \right)^2 dt \\
& \ll (QT)^{3-8c} \max_u \sum_{q \sim Q} \sum_{\chi \pmod{q}}^b \int_T^{2T} \left| \sum_{n > Q^2 T^2} \frac{\tau_4(n) \overline{\chi}(n)}{n^{5/4-c-it-iu}} \right|^2 dt \\
& \ll (QT)^{3-8c} \sum_{n > Q^2 T^2} (Q^2 T + n) \frac{\tau_4(n)^2}{n^{5/2-2c}} \ll (QT)^2 (\log(QT))^{16}.
\end{aligned}$$

Similarly

$$\begin{aligned}
& \sum_{q \sim Q} \sum_{\chi \pmod{q}}^b \int_T^{2T} |J_4(c, t, \chi)|^2 dt \\
& \ll (QT)^{-1-8c} \max_u \sum_{q \sim Q} \sum_{\chi \pmod{q}}^b \int_T^{2T} \left| \sum_{n \leq Q^2 T^2} \frac{\tau_4(n) \overline{\chi}(n)}{n^{1/4-c-it-iu}} \right|^2 dt \\
& \ll (QT)^{-1-8c} (Q^2 T + Q^2 T^2) \sum_{n \leq Q^2 T^2} \frac{\tau_4(n)^2}{n^{1/2-2c}} \ll (QT)^2 (\log(QT))^{16}.
\end{aligned}$$

□

In case of the Riemann zeta function, a better upper bound is available, and it will be helpful in evaluating the main terms:

Lemma 3.5. *Let $T \geq 3$. Then*

$$(16) \quad \int_{-T}^T |\zeta(1/2 + c + it)|^8 dt \ll T^{3/2} (\log T)^{21/2}$$

for any $c \geq 0$.

Proof. Let us first consider the case $c = 0$. By the Cauchy-Schwarz inequality

$$\int_{-T}^T |\zeta(1/2 + it)|^8 dt \ll \left(\int_{-T}^T |\zeta(1/2 + it)|^4 dt \right)^{1/2} \left(\int_{-T}^T |\zeta(1/2 + it)|^{12} dt \right)^{1/2}$$

Applying upper bounds for the fourth and twelfth power moments of the Riemann zeta function (see e.g. [12, formula (7.6.3)] for the fourth moment and see [6] for the twelfth moment), this is

$$\ll (T(\log T)^4)^{1/2} \cdot (T^2(\log T)^{17})^{1/2},$$

and the claim follows in case $c = 0$. In case $c \geq 1$ the left hand side of (16) is trivially bounded by $O(T)$. In the remaining case $c \in (0, 1)$ the claim follows from a convexity argument (see [12, Section 7.8]). \square

Let us here also record orthogonality relations for characters. There and later when \pm appears only on one side of an equation, both options are summed.

Lemma 3.6. *If m, n are integers with $(mn, q) = 1$ then*

$$\sum_{\chi \pmod{q}}^* \chi(m)\bar{\chi}(n) = \sum_{\substack{q=dr \\ r|(m-n)}} \mu(d)\phi(r),$$

and

$$\sum_{\chi \pmod{q}}^b \chi(m)\bar{\chi}(n) = \frac{1}{2} \sum_{\substack{q=dr \\ r|(m\pm n)}} \mu(d)\phi(r).$$

Proof. The first claim follows from the orthogonality of all characters and Möbius inversion, while the second claim follows from the first by detecting even characters with $\frac{1+\chi(-1)}{2}$. \square

4. THE APPROXIMATE FUNCTIONAL EQUATION AND TRUNCATION

Writing

$$G(s, t) := \Gamma^4\left(\frac{s}{2} + \frac{it}{2}\right) \Gamma^4\left(\frac{s}{2} - \frac{it}{2}\right)$$

we would like to find an approximation to

$$\int_{-\infty}^{\infty} |\Lambda(\tfrac{1}{2} + it, \chi)|^8 dt = \int_{-\infty}^{\infty} G(\tfrac{1}{2}, t) L^4(\tfrac{1}{2} + it, \chi) L^4(\tfrac{1}{2} - it, \bar{\chi}) dt.$$

Note that, for $\text{Re}(s) > 1$, we have

$$L^4(s + it, \chi) L^4(s - it, \bar{\chi}) = \sum_{m, n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{m^s n^s} \chi(m)\bar{\chi}(n) \left(\frac{n}{m}\right)^{it}.$$

For stating an approximate functional equation for the eighth moment, we need the weight function $V: \mathbb{R}_+^3 \rightarrow \mathbb{C}$ defined by

$$(17) \quad V(\xi, \eta; \mu) := \int_{-\infty}^{\infty} \left(\frac{\eta}{\xi}\right)^{it} W\left(\frac{\xi\eta\pi^4}{\mu^4}, t\right) dt$$

with $W: \mathbb{R}_+ \times \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$(18) \quad W(x, t) := \frac{1}{2\pi i} \int_{(1)} G(1/2 + s, t) x^{-s} \frac{ds}{s}.$$

For the short proof of the following proposition, see [1, Lemma 2.1] (but note that the definition of $P(\chi, t)$ there is missing a factor $(n/m)^{it}$).

Lemma 4.1. *Let χ be an even primitive character (mod q). Then*

$$(19) \quad \int_{-\infty}^{\infty} |\Lambda(1/2 + iy, \chi)|^8 dy = 2 \sum_{m,n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \chi(m)\overline{\chi}(n)V(m, n; q).$$

The integration in y gives rise to $V(m, n; q)$ on the right hand side of (19) which makes the essential support of m and n sums more restricted; from the following lemma, we see that the main contribution comes from when m, n are both at most $q^{2+\varepsilon}$.

Lemma 4.2. *The weight function $W(x, t)$ defined in (18) is a smooth function of $x \in (0, \infty)$. Furthermore the function $V(\xi, \eta; \mu)$ defined in (17) satisfies, for any $\xi, \eta, \mu > 0$ and any non-negative integers ν_1, ν_2, ν_3 ,*

$$(20) \quad \frac{d^{\nu_1} d^{\nu_2} d^{\nu_3}}{d\xi^{\nu_1} d\eta^{\nu_2} d\mu^{\nu_3}} V(\xi, \eta; \mu) \ll_{\nu_1, \nu_2, \nu_3} \exp\left(-\frac{\max(\xi, \eta)^{1/2}}{\mu}\right) \cdot \frac{1}{\xi^{\nu_1} \eta^{\nu_2} \mu^{\nu_3}}.$$

Proof. The proof is essentially the same as [4, Proof of Lemma 1] but for completeness we prove (20) here. Without loss of generality, we can assume that $\eta \geq \xi$. By definition, for any $c > 0$,

$$V(\xi, \eta; \mu) = \int_{-\infty}^{\infty} \left(\frac{\eta}{\xi}\right)^{it} \frac{1}{2\pi i} \int_{(c)} G(1/2 + s, t) \left(\frac{\xi\eta\pi^4}{\mu^4}\right)^{-s} \frac{ds}{s} dt.$$

Substituting $it = z$, we obtain

$$V(\xi, \eta; \mu) = -\frac{1}{2\pi} \int_{-i\infty}^{i\infty} \left(\frac{\eta}{\xi}\right)^z \int_{(c)} G(1/2 + s, -iz) \left(\frac{\xi\eta\pi^4}{\mu^4}\right)^{-s} \frac{ds}{s} dz.$$

We move the z -integration to the line $\operatorname{Re} z = -c$. Writing $z = -c + it$ and $s = c + iu$, we obtain

$$V(\xi, \eta; \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\eta}{\xi}\right)^{-c+it} \int_{-\infty}^{\infty} G(1/2 + c + iu, t - ic) \left(\frac{\xi\eta\pi^4}{\mu^4}\right)^{-c-iu} \frac{du}{c + iu} dt.$$

Taking derivatives, we see that

$$\begin{aligned} \frac{d^{\nu_1} d^{\nu_2} d^{\nu_3}}{d\xi^{\nu_1} d\eta^{\nu_2} d\mu^{\nu_3}} V(\xi, \eta; \mu) &\ll_{\nu_1, \nu_2, \nu_3} \frac{1}{\xi^{\nu_1} \eta^{\nu_2} \mu^{\nu_3}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G(1/2 + c + iu, t - ic)| \left(\pi \frac{\eta^{1/2}}{\mu}\right)^{-4c} \\ &\quad \cdot |t + u|^{\nu_1} |2c + i(u - t)|^{\nu_2} |c + iu|^{\nu_3} \frac{du}{|c| + |u|} dt. \end{aligned}$$

When $\eta^{1/2}/\mu \leq 10$ we take $c = 2$ and the result follows immediately. Otherwise we choose $c = \eta^{1/2}/\mu > 10$. Then, by Stirling's formula (12),

(21)

$$\begin{aligned} |G(1/2 + c + iu, t + ic)| \left(\pi \frac{\eta^{1/2}}{\mu}\right)^{-4c} &= \left| \Gamma\left(\frac{1}{4} + i\frac{u+t}{2}\right)^4 \Gamma\left(\frac{1}{4} + c + i\frac{u-t}{2}\right)^4 \right| (\pi c)^{-4c} \\ &\ll \exp(-|u+t|) \frac{|\frac{1}{4} + c + i\frac{u-t}{2}|^{4c}}{(e\pi c)^{4c}} \exp\left(-2|u-t| \arctan\left(\frac{|u-t|}{2c+1/2}\right)\right) \end{aligned}$$

If $|u - t| \leq 2c$, then (21) is

$$\ll \exp(-|u + t|) \frac{(2c)^{4c}}{(e\pi c)^{4c}} \ll \exp(-|u + t| - 4c) \ll \exp(-|u + t| - |u - t| - 2c).$$

On the other hand if $|u - t| > 2c$, then (21) is

$$\begin{aligned} &\ll \exp(-|u + t|) \frac{|u - t|^{4c}}{(e\pi c)^{4c}} \exp\left(-2|u - t| \cdot \frac{3}{4}\right) \\ &\ll \exp(-|u + t| - |u - t|) \exp\left(-c \left(4 \log(e\pi) + \frac{1}{2} \cdot \frac{|u - t|}{c} - 4 \log \frac{|u - t|}{c}\right)\right) \\ &\ll \exp(-|u + t| - |u - t| - 2c). \end{aligned}$$

Hence in any case

$$\frac{d^{\nu_1} d^{\nu_2} d^{\nu_3}}{d\xi^{\nu_1} d\eta^{\nu_2} d\mu^{\nu_3}} V(\xi, \eta; \mu) \ll_{\nu_1, \nu_2, \nu_3} \frac{e^{-c}}{\xi^{\nu_1} \eta^{\nu_2} \mu^{\nu_3}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-\frac{1}{2}(|u+t|+|u-t|)) dudt$$

and the claim follows. \square

Let us make some initial reductions to the left hand side of our claim (2). First, by Lemma 4.1

$$\sum_q \Psi\left(\frac{q}{Q}\right) \sum_{\chi \pmod{q}}^b \int_{-\infty}^{\infty} |\Lambda(\frac{1}{2} + iy, \chi)|^8 dy = 2\Delta(\Psi, Q),$$

where

$$\Delta(\Psi, Q) = \sum_q \Psi\left(\frac{q}{Q}\right) \sum_{\chi \pmod{q}}^b \sum_{m, n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \chi(m)\bar{\chi}(n) V(m, n; q).$$

The next step is to slightly truncate the sums over m and n in $\Delta(\Psi, Q)$. This truncation will allow us to apply the complementary divisor trick to reduce the conductor in Section 6. For the truncation, let $\varepsilon_0 > 0$ and define

$$(22) \quad Q_0 := \exp((\log Q)^{\varepsilon_0})$$

and

$$\tilde{\Delta}(\Psi, Q) = \sum_q \sum_{\chi \pmod{q}}^b \Psi\left(\frac{q}{Q}\right) \sum_{m, n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \chi(m)\bar{\chi}(n) V\left(m, n; \frac{q}{Q_0}\right).$$

Note that we expect (and will later show) that $\Delta(\Psi, Q) \asymp Q^2(\log Q)^{16}$. In this section we recall from [1] that $\tilde{\Delta}(\Psi, Q)$ is a sufficiently close approximation to $\Delta(\Psi, Q)$. This type of procedure has previously appeared in the contexts of moments also in other situations (see for instance the works of Soundararajan [10] and Soundararajan and Young [11]).

Proposition 4.3. *With the above notation*

$$\Delta(\Psi, Q) - \tilde{\Delta}(\Psi, Q) \ll_{\alpha} Q^2(\log Q)^{15+\varepsilon_0}.$$

Proof. This follows as [1, Proof of Proposition 3.1]. The key ingredient in the proof is the large sieve. In [1] one has $(\log Q)^\alpha$ in place of Q_0 but the same proof works by replacing every occurrence of $(\log Q)^\alpha$ by Q_0 — our choice of Q_0 is sufficiently small since in the critical case of [1, Proof of Proposition 3.1] where $Q^2/Q_0^2 \leq M, N \leq Q^2$ the parameters M and N are each summed over $\ll \log Q_0 \ll (\log Q)^{\varepsilon_0}$ dyadic intervals. \square

5. SPLITTING $\tilde{\Delta}(\Psi, Q)$

Now, by Proposition 4.3 and orthogonality of characters (see Lemma 3.6), it is sufficient to consider

$$\tilde{\Delta}(\Psi, Q) = \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d,r \\ (dr, mn)=1 \\ r|m \pm n}} \phi(r)\mu(d)\Psi\left(\frac{dr}{Q}\right) V\left(m, n, \frac{dr}{Q_0}\right).$$

Let $D = (\log Q)^{\Delta_0}$ for some Δ_0 to be determined later and split

$$\tilde{\Delta}(\Psi, Q) = \mathcal{D}(\Psi, Q) + \mathcal{S}_1(\Psi, Q) + \mathcal{O}(\Psi, Q),$$

where the diagonal term $\mathcal{D}(\Psi, Q)$ consists of the terms with $m = n$, the term $\mathcal{S}_1(\Psi, Q)$ consists of the remaining terms with $d > D$ and $\mathcal{O}(\Psi, Q)$ consists of the remaining terms with $d \leq D$.

The first two terms $\mathcal{D}(\Psi, Q)$ and $\mathcal{S}_1(\Psi, Q)$ were handled in [1, Propositions 4.1 and 5.1], with $(\log Q)^\alpha$ in place of Q_0 . The same arguments give the following asymptotic formulas.

Lemma 5.1. *Let $\varepsilon > 0$ and let $\tilde{\Psi}$ be the Mellin transform of Ψ , which is defined by*

$$(23) \quad \tilde{\Psi}(s) = \int_0^\infty \Psi(u) u^s \frac{du}{u}.$$

Then

$$\begin{aligned} \mathcal{D}(\Psi, Q) &= 2^{16} Q^2 \frac{(\log Q)^{16}}{16!} \tilde{\Psi}(2) \frac{\mathcal{A}}{2} \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{\mathcal{B}_p} \left(\frac{1}{p} - \frac{1}{p^2} - \frac{1}{p^3}\right)\right) \int_{-\infty}^{\infty} G(1/2, t) dt \\ &\quad + O_\varepsilon(Q^2 (\log Q)^{15+\varepsilon}), \end{aligned}$$

where

$$\mathcal{B}_p = \sum_{r=0}^{\infty} \frac{\tau_4^2(p^r)}{p^r} \quad \text{and} \quad \mathcal{A} = \prod_p \mathcal{B}_p \left(1 - \frac{1}{p}\right)^{16}.$$

Lemma 5.2. *Let $\varepsilon > 0$. There exists an absolute constant C such that*

$$\mathcal{S}_1(\Psi, Q) = \mathcal{M}\mathcal{S}_1(\Psi, Q) + O_\varepsilon\left(\frac{Q^2 (\log Q)^C}{D^{1-\varepsilon}}\right),$$

where

$$\mathcal{MS}_1(\Psi, Q) := - \sum_{\substack{m, n=1 \\ m \neq n}} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{(q, mn)=1} \Psi\left(\frac{q}{Q}\right) \left(\sum_{\substack{dr=q \\ d \leq D}} \mu(d) \right) V\left(m, n; \frac{q}{Q_0}\right).$$

In [1] the remaining term $\mathcal{O}(\Psi, Q)$ was treated by the complementary divisor trick. In doing this, the first step is to replace the arithmetic factor $\phi(r)$ by a smooth function which can be done by writing $\phi(r) = \sum_{al=r} \mu(a)l$ so that

$$\mathcal{O}(\Psi, Q) = \frac{1}{2} \sum_{\substack{m, n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d \leq D, \\ a, l \\ (dal, mn)=1 \\ al|m \pm n}} \mu(a)\mu(d)l \Psi\left(\frac{adl}{Q}\right) V\left(m, n, \frac{adl}{Q_0}\right).$$

It turns out that we can only use the complementary divisor trick when a is not too large (this is ultimately due to Proposition 3.2 involving a q -average, as explained in Remark 6.5 below). Accordingly we write

$$\mathcal{O}(\Psi, Q) = \mathcal{S}_2(\Psi, Q) + \mathcal{G}(\Psi, Q),$$

where $\mathcal{S}_2(\Psi, Q)$ consists of the terms with $a > A := \exp((\log Q)^{\varepsilon_0/2})$ and $\mathcal{G}(\Psi, Q)$ consists of the terms with $a \leq A$.

To handle $\mathcal{S}_2(\Psi, Q)$, we prove the following lemma.

Lemma 5.3. *Let $\varepsilon > 0$. One has*

$$\mathcal{S}_2(\Psi, Q) = \mathcal{MS}_2(\Psi, Q) + O_\varepsilon\left(\frac{Q^2 \exp(2(\log Q)^{\varepsilon/3})}{A^{\varepsilon/3}}\right),$$

where

$$\mathcal{MS}_2(\Psi, Q) := \sum_{\substack{m, n=1 \\ m \neq n}} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{(q, mn)=1} \Psi\left(\frac{q}{Q}\right) \left(\sum_{\substack{adl=q \\ a > A, d \leq D}} \frac{\mu(a)\mu(d)l}{\phi(al)} \right) V\left(m, n; \frac{q}{Q_0}\right).$$

Proof. The proof mostly follows the proof of [1, Proposition 5.1] (i.e. proof of Lemma 5.2). Recall that

$$\mathcal{S}_2(\Psi, Q) = \frac{1}{2} \sum_{\substack{m, n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{a > A, d \leq D, l \geq 1 \\ (dal, mn)=1 \\ al|m \pm n}} \mu(a)\mu(d)l \Psi\left(\frac{adl}{Q}\right) V\left(m, n, \frac{adl}{Q_0}\right).$$

We express the condition $al|m \pm n$ using the even characters $\chi \pmod{al}$. Hence

$$\mathcal{S}_2(\Psi, Q) = \sum_{\substack{a>A, \\ d \leq D, \\ l \geq 1}} \frac{\mu(a)\mu(d)l}{\phi(al)} \Psi\left(\frac{adl}{Q}\right) \sum_{\substack{\chi \pmod{al} \\ \chi(-1)=1}} \sum_{\substack{m,n=1 \\ m \neq n \\ (mn,d)=1}}^{\infty} \frac{\chi(m)\bar{\chi}(n)\tau_4(m)\tau_4(n)}{\sqrt{mn}} V\left(m, n, \frac{adl}{Q_0}\right).$$

The principal character $\chi = \chi_0$ gives the claimed main term, so we can concentrate on the contribution of the non-principal characters.

Reintroducing the terms with $m = n$ leads to (here our treatment is simpler than that in [1, Proof of Proposition 5.1]) an error at most

$$(24) \quad \sum_{n=1}^{\infty} \frac{\tau_4^2(n)}{n} \sum_{\substack{a>A \\ d \leq D, l \geq 1 \\ (dal,n)=1}} l \Psi\left(\frac{adl}{Q}\right) \left| V\left(n, n, \frac{adl}{Q_0}\right) \right|.$$

The support of Ψ is contained in $[1, 2]$, so by Lemma 4.2,

$$\sum_{n>Q^3} \frac{\tau_4^2(n)}{n} \sum_{\substack{a>A \\ d \leq D, l \geq 1 \\ (dal,n)=1}} l \Psi\left(\frac{adl}{Q}\right) \left| V\left(n, n, \frac{adl}{Q_0}\right) \right| \ll Q^2 \sum_{n>Q^3} \exp\left(\frac{-n^{1/2}}{2Q}\right) \ll \frac{1}{Q},$$

so the quantity in (24) is

$$\ll \sum_{n \leq Q^3} \frac{\tau_4^2(n)}{n} \sum_{A < a \leq 2Q} \sum_{d \leq D} \sum_{l \leq \frac{2Q}{ad}} l + \frac{1}{Q} \ll \frac{Q^2 (\log Q)^{16}}{A}.$$

Hence it suffices to bound the sum

$$\mathcal{S}'_2(\Psi, Q) = \sum_{\substack{a>A, \\ d \leq D, \\ l \geq 1}} \frac{\mu(a)\mu(d)l}{\phi(al)} \Psi\left(\frac{adl}{Q}\right) \sum_{\substack{\chi \pmod{al} \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \sum_{\substack{m,n=1 \\ (mn,d)=1}}^{\infty} \frac{\chi(m)\bar{\chi}(n)\tau_4(m)\tau_4(n)}{\sqrt{mn}} V\left(m, n, \frac{adl}{Q_0}\right)$$

Arguing similarly to the proof of Proposition 5.1 in [1] but with al in place of r , with not reducing to primitive characters, and with an additional factor $\mu(a)l/\phi(al) \ll \log Q/\phi(a)$, we get that

$$(25) \quad \mathcal{S}'_2(\Psi, Q) \ll_{\varepsilon} (\log Q)^2 \sum_{d \leq D} d^{\varepsilon} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-|t_1| - |t_2|) \sum_{a>A} \frac{1}{\phi(a)} \sum_{l \leq \frac{2Q}{ad}} 1 \\ \cdot \sum_{\substack{\chi \pmod{al} \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \left\{ \left| L\left(\frac{1}{2} + \frac{1}{\log Q} + it_1, \chi\right) \right|^8 + \left| L\left(\frac{1}{2} + \frac{1}{\log Q} + it_2, \bar{\chi}\right) \right|^8 \right\} dt_1 dt_2.$$

Now, writing $al = r$,

$$\sum_{a>A} \frac{1}{\phi(a)} \sum_{l \leq 2Q/(ad)} 1 \leq \sum_{r \leq 2Q/d} \sum_{\substack{a|r \\ a>A}} \frac{1}{\phi(a)}.$$

Here

$$(26) \quad \sum_{\substack{a|r \\ a>A}} \frac{1}{\phi(a)} \ll \frac{\log Q}{A^{\varepsilon/3}} \sum_{a|r} \frac{1}{a^{1-\varepsilon/3}} \ll \frac{\log Q}{A^{\varepsilon/3}} \prod_{p|r} \left(1 + \frac{1}{p^{1-\varepsilon/3}}\right) \\ \ll \frac{\log Q}{A^{\varepsilon/3}} \prod_{p \leq \log Q} \left(1 + \frac{1}{p^{1-\varepsilon/3}}\right) \ll \frac{\log Q}{A^{\varepsilon/3}} \exp((\log Q)^{\varepsilon/3}),$$

Using this and Proposition 3.2 in (25), we get

$$\mathcal{S}'_2(\Psi, Q) \ll_{\varepsilon} \frac{\exp(\frac{3}{2}(\log Q)^{\varepsilon/3})}{A^{\varepsilon/3}} \sum_{d \leq D} d^{\varepsilon} \max_{R \leq \frac{2Q}{d}} \int_{-\infty}^{\infty} \exp(-|t_1|) \\ \cdot \sum_{r \sim R} \sum_{\substack{\chi \bmod r \\ \chi(-1)=1}} \left| L\left(\frac{1}{2} + \frac{1}{\log Q} + it_1, \chi\right) \right|^8 dt_1 \\ \ll_{\varepsilon} \frac{\exp(\frac{3}{2}(\log Q)^{\varepsilon/3})}{A^{\varepsilon/3}} \sum_{d \leq D} d^{\varepsilon} \max_{R \leq \frac{2Q}{d}} R^2 (\log Q)^{16} \ll \frac{Q^2 \exp(2(\log Q)^{\varepsilon/3})}{A^{\varepsilon/3}},$$

as claimed. \square

Remark 5.4. The bound (26) is the reason we need to make a rather large choice like $A = \exp((\log Q)^{\varepsilon_0/2})$ which in turn in the treatment of $\mathcal{G}(\Psi, Q)$ will force a choice like $Q_0 = \exp((\log Q)^{\varepsilon_0})$.

Combining the previous two lemmas we obtain

Proposition 5.5. *Let $\varepsilon > 0$. There exists an absolute constant C such that*

$$\mathcal{S}_1(\Psi, Q) + \mathcal{S}_2(\Psi, Q) = \mathcal{MS}(\Psi, Q) + O_{\varepsilon} \left(\frac{Q^2 (\log Q)^C}{D^{1-\varepsilon}} + \frac{Q^2 \exp(2(\log Q)^{\varepsilon/3})}{A^{\varepsilon/3}} \right),$$

where

(27)

$$\mathcal{MS}(\Psi, Q) := - \sum_{\substack{m, n=1 \\ m \neq n}} \frac{\tau_4(m) \tau_4(n)}{\sqrt{mn}} \sum_{(q, mn)=1} \Psi\left(\frac{q}{Q}\right) \left(\sum_{\substack{adl=q \\ a \leq A, d \leq D}} \frac{\mu(a) \mu(d) l}{\phi(al)} \right) V\left(m, n; \frac{q}{Q_0}\right).$$

Proof. Writing the sum over d in $\mathcal{MS}_1(\Psi, Q)$ as

$$\sum_{\substack{dr=q \\ d \leq D}} \mu(d) = \sum_{\substack{dr=q \\ d \leq D}} \mu(d) \sum_{r=al} \frac{\mu(a) l}{\phi(r)} = \sum_{\substack{adl=q \\ d \leq D}} \frac{\mu(a) \mu(d) l}{\phi(al)}$$

we see that

$$\mathcal{MS}_1(\Psi, Q) + \mathcal{MS}_2(\Psi, Q) = \mathcal{MS}(\Psi, Q),$$

so the claim follows from Lemmas 5.2 and 5.3. \square

We have now understood $\mathcal{D}(\Psi, Q)$ and $\mathcal{S}_1(\Psi, Q) + \mathcal{S}_2(\Psi, Q)$, and we will turn our attention to $\mathcal{G}(\Psi, Q)$ in the following section.

6. TREATMENT OF $\mathcal{G}(\Psi, Q)$

6.1. The complementary divisor. Recall that

$$(28) \quad \mathcal{G}(\Psi, Q) = \frac{1}{2} \sum_{\substack{m, n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{a \leq A, d \leq D, l \geq 1 \\ (dal, mn)=1 \\ al | m \pm n}} \mu(a)\mu(d)l\Psi\left(\frac{adl}{Q}\right) V\left(m, n, \frac{adl}{Q_0}\right).$$

We write $g = (m, n)$ and $m = gM$, $n = gN$, so that $(M, N) = 1$. Note that the condition $(al, mn) = 1$ can be replaced by $(al, g) = 1$ since $al | m \pm n$. Necessarily $al | M \pm N$ and we write $|M \pm N| = alh$. We want to replace the condition modulo al with a condition modulo ah , which will be small when l is large. To do so, we express the condition $(l, g) = 1$ by $\sum_{b|(l, g)} \mu(b)$. Writing $l = bk$, the inner sum in (28) becomes

$$\sum_{\substack{d \leq D \\ (d, gMN)=1}} \mu(d) \sum_{\substack{a \leq A \\ (a, g)=1}} \mu(a) \sum_{b|g} \mu(b) \sum_{\substack{k \geq 1 \\ |M \pm N| = abkh}} bk\Psi\left(\frac{dabk}{Q}\right) V\left(gM, gN, \frac{dabk}{Q_0}\right).$$

Substituting $k = \frac{|M \pm N|}{abh}$ and rearranging, this equals

$$(29) \quad Q \sum_{\substack{d \leq D \\ (d, gMN)=1}} \sum_{\substack{a \leq A \\ (a, g)=1}} \sum_{b|g} \sum_{\substack{h > 0 \\ M \equiv \mp N \pmod{abh}}} \frac{\mu(a)\mu(b)\mu(d)}{ad} \cdot \left(\frac{d|M \pm N|}{Qh}\right) \Psi\left(\frac{d|M \pm N|}{Qh}\right) V\left(gM, gN; \frac{d|M \pm N|}{hQ_0}\right).$$

For non-negative real numbers u, x, y and for each choice of sign, we define

$$(30) \quad \mathcal{W}^{\pm}(x, y; u) = u|x \pm y|\Psi(u|x \pm y|)V(x, y; u|x \pm y|),$$

and

$$\mathcal{W}(x, y; u) = \mathcal{W}^+(x, y; u) + \mathcal{W}^-(x, y; u).$$

It is immediate from the definition of $V(m, n; \mu)$ that, for any $c > 0$,

$$(31) \quad V(cm, cn; \sqrt{c}\mu) = \int_{-\infty}^{\infty} \left(\frac{n}{m}\right)^{it} W\left(\frac{mn\pi^4}{\mu^4}, t\right) dt = V(m, n; \mu).$$

Thus

$$\mathcal{G}(\Psi, Q) = \frac{Q}{2} \sum_{\pm} \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d \leq D \\ (d, gMN)=1}} \sum_{\substack{a \leq A \\ (a, g)=1}} \sum_{b|g} \sum_{\substack{h > 0 \\ M \equiv \mp N \pmod{abh}}} \frac{\mu(a)\mu(b)\mu(d)}{ad} \\ \cdot \mathcal{W}^{\pm} \left(\frac{gMQ_0^2}{Q^2}, \frac{gNQ_0^2}{Q^2}, \frac{Qd}{ghQ_0^2} \right),$$

where \sum_{\pm} indicates the sum over both choices of signs \pm . Note that since $(M, N) = 1$, necessarily $(MN, abh) = 1$.

We express the condition $M \equiv \mp N \pmod{abh}$ using characters $\chi \pmod{abh}$. We then separate the principal character contribution, which is the main term. Specifically, we write

$$\mathcal{G}(\Psi, Q) = \mathcal{MG}(\Psi, Q) + \mathcal{EG}(\Psi, Q),$$

where

$$(32) \quad \mathcal{MG}(\Psi, Q) := \frac{Q}{2} \sum_{\pm} \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d \leq D \\ (d, gMN)=1}} \sum_{\substack{a \leq A \\ (a, gMN)=1}} \sum_{\substack{b|g \\ (b, MN)=1}} \sum_{\substack{h > 0 \\ (h, MN)=1}} \frac{\mu(a)\mu(b)\mu(d)}{ad\phi(abh)} \\ \cdot \mathcal{W}^{\pm} \left(\frac{gMQ_0^2}{Q^2}, \frac{gNQ_0^2}{Q^2}, \frac{dQ}{ghQ_0^2} \right),$$

and

$$(33) \quad \mathcal{EG}(\Psi, Q) := \frac{Q}{2} \sum_{\pm} \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d \leq D \\ (d, gMN)=1}} \sum_{\substack{a \leq A \\ (a, gMN)=1}} \sum_{\substack{b|g \\ (b, MN)=1}} \sum_{\substack{h > 0 \\ (h, MN)=1}} \frac{\mu(a)\mu(b)\mu(d)}{ad\phi(abh)} \\ \cdot \sum_{\substack{\chi \pmod{abh} \\ \chi \neq \chi_0}} \chi(M)\bar{\chi}(\mp N) \mathcal{W}^{\pm} \left(\frac{gMQ_0^2}{Q^2}, \frac{gNQ_0^2}{Q^2}, \frac{dQ}{ghQ_0^2} \right).$$

6.2. Mellin transforms of \mathcal{W}^{\pm} . To evaluate $\mathcal{MG}(\Psi, Q)$ and $\mathcal{EG}(\Psi, Q)$, we will write $\mathcal{W}^{\pm}(x, y; u)$ in terms of its Mellin transforms. We shall consider three different types of Mellin transforms. They come from taking Mellin transforms in the variable u when we need to sum over the modulus h , the variables x and y when we need to sum over M and N , and in all three variables when we need to sum over M , N , and h . (In the description above, we have neglected to mention the conceptually less important sums over d , a , b and g .)

We collect the properties of the various Mellin transforms in the following three lemmas. The first lemma is from [1, Section 6.2] and the proof is the same as in [4, Section 7], but using the bounds of Lemma 4.2 in place of [4, Lemma 1].

Lemma 6.1. *Given positive real numbers x and y , define*

$$\widetilde{\mathcal{W}}_1^\pm(x, y; z) = \int_0^\infty \mathcal{W}^\pm(x, y; u) u^z \frac{du}{u}.$$

Then the functions $\widetilde{\mathcal{W}}_1^\pm(x, y; z)$ are analytic for all $z \in \mathbb{C}$. For any $c \in \mathbb{R}$, we have the Mellin inversion formula

$$\mathcal{W}^\pm(x, y; u) = \frac{1}{2\pi i} \int_{(c)} \widetilde{\mathcal{W}}_1^\pm(x, y; z) u^{-z} dz.$$

For any non-negative integer ν , any real numbers $x, y > 0$ and any $z \in \mathbb{C}$ one has

$$|\widetilde{\mathcal{W}}_1^\pm(x, y; z)| \ll_\nu |x \pm y|^{-\operatorname{Re} z} \prod_{j=1}^\nu |z + j|^{-1} \exp\left(-c_1 \max(x, y)^{1/4}\right)$$

for some absolute constant c_1 .

The next lemma is similar to Lemma 6.3 in [1], but we include a bound for the error resulting from truncating the integrals over s_1 and s_2 . In both [1] and [4], the truncation was not explained. We include it here for completeness.

Lemma 6.2. *Define*

$$\widetilde{\mathcal{W}}_3^\pm(s_1, s_2; z) = \int_0^\infty \int_0^\infty \int_0^\infty \mathcal{W}^\pm(x, y; u) u^z x^{s_1} y^{s_2} \frac{du}{u} \frac{dx}{x} \frac{dy}{y},$$

and

$$\widetilde{\mathcal{W}}_3(s_1, s_2; z) = \widetilde{\mathcal{W}}_3^+(s_1, s_2; z) + \widetilde{\mathcal{W}}_3^-(s_1, s_2; z).$$

Let $\omega = \frac{s_1 + s_2 - z}{2}$ and $\xi = \frac{s_1 - s_2 + z}{2}$. For $\operatorname{Re}(s_1), \operatorname{Re}(s_2) > 0$, and $|\operatorname{Re}(s_1 - s_2)| < \operatorname{Re}(z) < 1$ we have

$$(34) \quad \widetilde{\mathcal{W}}_3(s_1, s_2; z) = \frac{\widetilde{\Psi}(1 + 4\omega + z)}{2\omega\pi^{4\omega}} \int_{-\infty}^\infty \mathcal{H}(\xi - it, z) G\left(\frac{1}{2} + \omega, t\right) dt,$$

where $\widetilde{\Psi}$ is defined in (23), and

$$(35) \quad \begin{aligned} \mathcal{H}(u, v) &= \frac{\Gamma(u)\Gamma(v-u)}{\Gamma(v)} + \frac{\Gamma(u)\Gamma(1-v)}{\Gamma(1+u-v)} + \frac{\Gamma(v-u)\Gamma(1-v)}{\Gamma(1-u)} \\ &= \pi^{1/2} \frac{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{1-v}{2}\right)\Gamma\left(\frac{v-u}{2}\right)}{\Gamma\left(\frac{1-u}{2}\right)\Gamma\left(\frac{v}{2}\right)\Gamma\left(\frac{1-v+u}{2}\right)}. \end{aligned}$$

Let $x \neq y$ and $T \geq Q^\varepsilon$. For any $c_1, c_2 > 0$ with $|c_1 - c_2| < c < 1$, one has the truncated Mellin inversion formulas

(36)

$$\mathcal{W}(x, y; u) = \frac{1}{(2\pi i)^3} \int_{(c)} \int_{c_1 - iT}^{c_1 + iT} \int_{c_2 - iT}^{c_2 + iT} \widetilde{\mathcal{W}}_3(s_1, s_2; z) u^{-z} x^{-s_1} y^{-s_2} ds_2 ds_1 dz + O\left(\frac{u^{-c} x^{-c_1} y^{-c_2}}{T^{1-c} \left|\log\left(\frac{x}{y}\right)\right|}\right).$$

Moreover, let $\widetilde{\mathcal{W}}_1(x, y; z) = \widetilde{\mathcal{W}}_1^+(x, y; z) + \widetilde{\mathcal{W}}_1^-(x, y; z)$. Then for $\operatorname{Re} z = c$,

$$(37) \quad \widetilde{\mathcal{W}}_1(x, y; z) = \frac{1}{(2\pi i)^2} \int_{c_1 - iT}^{c_1 + iT} \int_{c_2 - iT}^{c_2 + iT} \widetilde{\mathcal{W}}_3(s_1, s_2; z) x^{-s_1} y^{-s_2} ds_2 ds_1 + O\left(\frac{x^{-c_1} y^{-c_2}}{T^{1-c} \left|\log\left(\frac{x}{y}\right)\right| (1 + |z|)^A}\right),$$

for any $A > 0$. Finally, for $\operatorname{Re}(s_1), \operatorname{Re}(s_2) > 0$, and $|\operatorname{Re}(s_1 - s_2)| < \operatorname{Re}(z) < 1$, the Mellin transform $\widetilde{\mathcal{W}}_3(s_1, s_2; z)$ satisfies the bound

$$(38) \quad |\widetilde{\mathcal{W}}_3(s_1, s_2; z)| \ll (1 + |z|)^{-A} (1 + |\omega|)^{-A} (1 + |\xi|)^{\operatorname{Re}(z) - 1},$$

for any $A > 0$.

Proof. All claims but (36) and (37) are from [4, Lemma 6]. By symmetry we can assume that $x < y$. By (34), Mellin inversion, and a change of variables,

(39)

$$\mathcal{W}(x, y; u) = \frac{1}{(2\pi i)^3} \int_{(b_1)} \int_{(c)} \int_{-\infty}^{\infty} u^{-z} \widetilde{\Psi}(1 + 4s + z) \int_{(b_3)} \mathcal{H}(w, z) \left(\frac{x}{y}\right)^{-w} dw \cdot G\left(\frac{1}{2} + s, t\right) \frac{y^{-s-z+it} x^{-s-it}}{\pi^{4s}} dt dz \frac{ds}{s}.$$

In the above, note that the integral in w does not converge absolutely, and we interpret this as a symmetrical limit - see (41) for the precise expression. Taking the Mellin transform of (39) as a function of u and using the definition of $\widetilde{\mathcal{W}}_1$ in Lemma 6.1, we obtain

$$(40) \quad \widetilde{\mathcal{W}}_1(x, y; z) = \frac{1}{(2\pi i)^2} \int_{(b_1)} \int_{-\infty}^{\infty} \widetilde{\Psi}(1 + 4s + z) \int_{(b_3)} \mathcal{H}(w, z) \left(\frac{x}{y}\right)^{-w} dw \cdot G\left(\frac{1}{2} + s, t\right) \frac{y^{-s-z+it} x^{-s-it}}{\pi^{4s}} dt \frac{ds}{s}.$$

We now give a proof of the truncated Mellin inversion formula (36), the proof of (37) following by the same lines. Here, we note that the rapid decay in z in the error term in (37) follows from the presence of the factor $\widetilde{\Psi}(1 + 4s + z)G\left(\frac{1}{2} + s, t\right)$.

Note that the integrals over t , z and s decay rapidly along vertical lines due to the factors $\tilde{\Psi}$ and $G\left(\frac{1}{2} + s, t\right)$. The integrals over t , z and s can thus be truncated to short integrals of length $T^{1/2} \geq Q^{\varepsilon/2}$, up to an error of

$$\frac{u^{-c} x^{-b_3-b_1} y^{-b_1-c+b_3}}{T^{100}}.$$

On the other hand, the integral in w requires more care as it is not absolutely convergent, and should be interpreted as a symmetrical limit. To be more precise

$$(41) \quad \frac{1}{2\pi i} \int_{(b_3)} \mathcal{H}(w, z) \left(\frac{x}{y}\right)^{-w} dw = \lim_{T \rightarrow \infty} \int_{b_3-iT}^{b_3+iT} \mathcal{H}(w, z) \left(\frac{x}{y}\right)^{-w} dw.$$

Now, consider z fixed with $|\operatorname{Im} z| \leq Q^{\varepsilon/2}$ and $x < y$. By Stirling's formula for Gamma functions in $\mathcal{H}(w, z)$, for $\operatorname{Re} z = c$, we have that

$$(42) \quad \left| \left(\int_{b_3+iT}^{-\infty+iT} + \int_{-\infty-iT}^{b_3-iT} \right) \mathcal{H}(w, z) \left(\frac{x}{y}\right)^{-w} dw \right| \ll \frac{1}{T^{1-c}} \int_{-\infty}^{b_3} \left(\frac{x}{y}\right)^{-\sigma} d\sigma \\ \ll \frac{x^{-b_3} y^{b_3}}{T^{1-c} \left| \log \left(\frac{x}{y}\right) \right|}.$$

In the above, whenever $\lambda = \nu \pm i\tau$ with $|\tau| \geq 100$ is not within the region allowed by Stirling's formula, we write $|\Gamma(\lambda)| = \frac{\pi}{|\Gamma(1-\lambda)\sin(\pi\lambda)|} \asymp \frac{1}{e^{\pi|\tau|} |\Gamma(1-\lambda)|}$, and apply Stirling's formula to $\Gamma(1-\lambda)$ instead. This and (41) gives that

$$(43) \quad \frac{1}{2\pi i} \int_{(b_3)} \mathcal{H}(w, z) \left(\frac{x}{y}\right)^{-w} dw \\ = \lim_{T \rightarrow \infty} \left(\int_{b_3-iT}^{b_3+iT} + \int_{b_3+iT}^{-\infty+iT} + \int_{-\infty-iT}^{b_3-iT} \right) \mathcal{H}(w, z) \left(\frac{x}{y}\right)^{-w} dw.$$

Moreover, by (35),

$$\left(\int_{b_3-iT}^{b_3+iT} + \int_{b_3+iT}^{-\infty+iT} + \int_{-\infty-iT}^{b_3-iT} \right) \mathcal{H}(w, z) \left(\frac{x}{y}\right)^{-w} dw \\ = \left(\int_{b_3-iT}^{b_3+iT} + \int_{b_3+iT}^{-\infty+iT} + \int_{-\infty-iT}^{b_3-iT} \right) \left[\frac{\Gamma(w)\Gamma(z-w)}{\Gamma(z)} + \frac{\Gamma(w)\Gamma(1-z)}{\Gamma(1+w-z)} \right. \\ \left. + \frac{\Gamma(z-w)\Gamma(1-z)}{\Gamma(1-w)} \right] \left(\frac{x}{y}\right)^{-w} dw \\ = \left| 1 + \frac{x}{y} \right|^{-z} + \left| 1 - \frac{x}{y} \right|^{-z},$$

upon summing the residues of the integrand within the region bounded by our contour of integration. Here, we have noted that the integrand above

goes to zero as $\operatorname{Re} w \rightarrow -\infty$ since $x < y$ and by Stirling's formula. Note that this is independent of T for $T \geq Q^\varepsilon$ and so (43) gives that

$$(44) \quad \left(\int_{b_3-iT}^{b_3+iT} + \int_{b_3+iT}^{-\infty+iT} + \int_{-\infty-iT}^{b_3-iT} \right) \mathcal{H}(w, z) \left(\frac{x}{y} \right)^{-w} dw = \frac{1}{2\pi i} \int_{(b_3)} \mathcal{H}(w, z) \left(\frac{x}{y} \right)^{-w} dw$$

Inserting the above and (42) into (39), we obtain that, for $x < y$ and $0 < b_3 < c < 1$,

$$(45) \quad \mathcal{W}(x, y; u) = \mathcal{I}(x, y, u; T) + O\left(\frac{u^{-c} y^{-b_1+b_3-c} x^{-b_1-b_3}}{T^{1-c} \left| \log \left(\frac{x}{y} \right) \right|} \right),$$

where

$$\begin{aligned} \mathcal{I}(x, y, u; T) = & \frac{1}{(2\pi i)^3} \int_{(b_1)} \int_{(c)} \int_{b_3-iT/2}^{b_3+iT/2} \int_{-\infty}^{\infty} u^{-z} \tilde{\Psi}(1+4s+z) \mathcal{H}(w, z) G\left(\frac{1}{2}+s, t\right) \\ & \times \frac{y^{-s+it+w-z} x^{-s-it-w}}{\pi^{4s}} dt dw dz \frac{ds}{s} \end{aligned}$$

Now we change the variable $s_1 = s + it + w$ and $s_2 = s - it - w + z$. Recall that $T \geq Q^\varepsilon$. Let ω and ξ be defined as in the statement of the lemma. We then have for $|c_1 - c_2| < c$

$$\begin{aligned} \mathcal{W}(x, y; u) = & \frac{1}{(2\pi i)^3} \int_{(c)} \int_{c_1-iT}^{c_1+iT} \int_{c_2-iT}^{c_2+iT} \tilde{\mathcal{W}}_3(s_1, s_2; z) u^{-z} y^{-s_2} x^{-s_1} ds_2 ds_1 dz \\ & + O\left(Q^{-1000} + \frac{u^{-c} y^{-c_2} x^{-c_1}}{T^{1-c} \left| \log \left(\frac{x}{y} \right) \right|} \right) \end{aligned}$$

as desired. \square

The following lemma is of crucial importance when we remove the need for GRH — in [1, 4] the decay in $|s_1 + s_2|$ was not noticed or utilized.

Lemma 6.3. *Given a positive real number u , we define*

$$\tilde{\mathcal{W}}_2^\pm(s_1, s_2; u) = \int_0^\infty \int_0^\infty \mathcal{W}^\pm(x, y; u) x^{s_1} y^{s_2} \frac{dx dy}{x y}.$$

Then the functions $\tilde{\mathcal{W}}_2^\pm(s_1, s_2; u)$ are analytic in the region $\operatorname{Re} s_1, \operatorname{Re} s_2 > 0$. We have the Mellin inversion formula

$$\mathcal{W}^\pm(x, y; u) = \frac{1}{(2\pi i)^2} \int_{(c_1)} \int_{(c_2)} \tilde{\mathcal{W}}_2^\pm(s_1, s_2; u) x^{-s_1} y^{-s_2} ds_2 ds_1,$$

when c_1 and c_2 are positive. For any integers $k \geq 1, l \geq 0$ and any $s_1, s_2 \in \mathbb{C}$ with $0 < \operatorname{Re} s_1, \operatorname{Re} s_2 \leq 100$, one has

$$(46) \quad |\tilde{\mathcal{W}}_2^\pm(s_1, s_2; u)| \ll_{k,l} \frac{1}{\operatorname{Re} s_1 \cdot \operatorname{Re} s_2} \cdot \frac{(1+u)^{k-1}}{\max\{|s_1|, |s_2|\}^k |s_1 + s_2|^l} \exp\left(-c'u^{-1/4}\right)$$

for some constant $c' > 0$.

Proof. Apart from (46) the claims go back to [4] (and are easy to prove). In the proof of (46) we can assume by symmetry that $|s_1| \geq |s_2|$. The proof is based on partial integration, and for this we first study the derivatives of $\mathcal{W}^\pm(x, y; u)$.

To simplify the notation, we write in this proof V_1, V_2, \dots for unspecified functions $V_j: \mathbb{R}_+^3 \rightarrow \mathbb{C}$ that satisfy (20) and W_1^\pm, W_2^\pm, \dots for unspecified functions $W_j^\pm: \mathbb{R}_+^3 \rightarrow \mathbb{C}$ that are of the form $W_j^\pm(x, y; u) = u|x \pm y| \Psi(u|x \pm y|) V_j(x, y; u|x \pm y|)$ for some such V_j .

By Lemma 4.2 and the chain rule

$$\frac{d}{dx} V(x, y; u|x \pm y|) = \frac{1}{x} V_1(x, y; u|x \pm y|) + u \cdot \frac{1}{u|x \pm y|} V_2(x, y; u|x \pm y|),$$

so that in the region $u|x \pm y| \in [1, 2]$, we have, for any $j \geq 0$

$$\frac{d^j}{dx^j} V(x, y; u|x \pm y|) = \left(\frac{1}{x^j} + u^j \right) V_3(x, y; u|x \pm y|)$$

and

$$\frac{d^j}{dx^j} \mathcal{W}^\pm(x, y; u|x \pm y|) = \left(\frac{1}{x^j} + u^j \right) W_1(x, y; u|x \pm y|).$$

Note that we avoid the non-differentiable point of $|x \pm y|$ due to the support of Ψ . Hence by partial integration k times, we see that

$$|\widetilde{\mathcal{W}}_2^\pm(s_1, s_2; u)| \ll \frac{1}{|s_1|^k} \left| \int_0^\infty \int_0^\infty (1+ux)^k W_1^\pm(x, y; u) x^{s_1} y^{s_2} \frac{dx}{x} \frac{dy}{y} \right|,$$

Now we substitute $x = w$ and $y = wz$, so that

$$(47) \quad |\widetilde{\mathcal{W}}_2^\pm(s_1, s_2; u)| \ll \frac{1}{|s_1|^k} \left| \int_0^\infty \int_0^\infty (1+uw)^k W_1^\pm(w, wz; u) w^{s_1+s_2} z^{s_2} \frac{dw}{w} \frac{dz}{z} \right|.$$

Next we perform partial integration with respect to w . For this note that, by Lemma 4.2 and the chain rule,

$$\begin{aligned} \frac{d}{dw} V(w, wz, uw|1 \pm z|) &= \frac{1}{w} V_4(w, wz, uw|1 \pm z|) + z \cdot \frac{1}{wz} V_5(w, wz, uw|1 \pm z|) \\ &\quad + u|1 \pm z| \frac{1}{uw|1 \pm z|} V_6(w, wz, uw|1 \pm z|) \\ &= \frac{1}{w} V_7(w, wz, uw|1 \pm z|). \end{aligned}$$

Note also that

$$\frac{d}{dw} (1+uw)^k = \frac{1}{w} (1+uw)^k \cdot k \frac{uw}{1+uw}.$$

Using similar bounds for higher order derivatives and considering other terms as well, we see that

$$\frac{d^l}{dw^l} \left((1+uw)^k W_1^\pm(w, wz; u) \right) = \frac{1}{w^l} (1+uw)^k W_2^\pm(w, wz; u).$$

Hence, by applying partial integration l times to the right hand side of (47), we obtain

$$|\widetilde{\mathcal{W}}_2^\pm(s_1, s_2; u)| \ll \frac{1}{|s_1|^k |s_1 + s_2|^l} \left| \int_0^\infty \int_0^\infty w^{s_1+s_2} z^{s_2} (1+uw)^k W_3^\pm(w, wz; u) \frac{dz}{z} \frac{dw}{w} \right|.$$

Substituting back $w = x$ and $z = y/x$, we see that the above integral is

$$(48) \quad \int_0^\infty \int_0^\infty x^{s_1} y^{s_2} (1+ux)^k W_3^\pm(x, y; u) \frac{dy}{y} \frac{dx}{x}.$$

Recall that $k \geq 1$ and $\operatorname{Re} s_i > 0$ and that $W_3^\pm(x, y; u)$ is supported on $u|x \pm y| \in [1, 2]$ and satisfies $W_3^\pm(x, y; u) \ll \exp(-2c \max\{x, y\}^{1/4})$ for some $c > 0$.

In the part of (48) with $x > 10/u$, we have $y \asymp x$ and y is restricted to an interval of length $1/u$ (depending on x). Hence the contribution of $x > 10/u$ to (48) is bounded by

$$\begin{aligned} & u^{k-1} \int_{10/u}^\infty x^{\operatorname{Re} s_1 + \operatorname{Re} s_2} x^k \exp(-2cx^{1/4}) \frac{dx}{x^2} \\ & \ll u^{k-1} \frac{1}{\operatorname{Re} s_1 + \operatorname{Re} s_2 + k - 1} \left(\frac{1}{u^{\operatorname{Re} s_1 + \operatorname{Re} s_2 + k - 1}} + 1 \right) \exp(-2cu^{-1/4}) \\ & \ll u^{k-1} \frac{1}{\operatorname{Re} s_1 \cdot \operatorname{Re} s_2} \exp(-cu^{-1/4}). \end{aligned}$$

In the part of (48) with $x \leq 10/u$, we have $y \leq 12/u$ and $\max\{x, y\} \geq 1/(2u)$ (since $|x \pm y| \in [1/u, 2/u]$), so the contribution of this part is bounded by

$$\begin{aligned} & \int_0^{10/u} \int_0^{12/u} \mathbf{1}_{\max\{x, y\} \geq 1/(2u)} x^{\operatorname{Re} s_1} y^{\operatorname{Re} s_2} |W_3^\pm(x, y; u)| \frac{dx}{x} \frac{dy}{y} \\ & \ll \frac{1}{\operatorname{Re} s_1} \frac{1}{\operatorname{Re} s_2} u^{-\operatorname{Re} s_1 - \operatorname{Re} s_2} \exp(-2cu^{-1/4}) \ll \frac{1}{\operatorname{Re} s_1} \frac{1}{\operatorname{Re} s_2} \exp(-cu^{-1/4}). \end{aligned}$$

□

6.3. Bounding the error term $\mathcal{E}\mathcal{G}(\Psi, Q)$. The aim of this section is to prove the following.

Lemma 6.4. *Let $\mathcal{E}\mathcal{G}(\Psi, Q)$ be as in (33) with*

$$(49) \quad A = \exp((\log Q)^{\varepsilon_0/2}), \quad D = (\log Q)^{\Delta_0}, \quad \text{and} \quad Q_0 = \exp((\log Q)^{\varepsilon_0}),$$

where we recall that ε_0 and Δ_0 are fixed positive constants to be chosen later. Then

$$\mathcal{E}\mathcal{G}(\Psi, Q) \ll_C \frac{Q^2}{(\log Q)^C}$$

for any $C \geq 1$.

Proof. Following [1, Proof of Lemma 7.1] until [1, (25)], we see that $\mathcal{EG}(\Psi, Q)$ is

$$(50) \quad \ll Q(\log Q)^{16} \sum_{\pm} \sum_{\substack{a \leq A \\ b, h > 0}} \sum_{\substack{\chi \pmod{abh} \\ \chi \neq \chi_0}} \sum_{\substack{g \\ b|g, (a, g)=1}} \sum_{\substack{d \leq D \\ (d, g)=1}} \frac{\tau^3(d)\tau^3(g)\tau_4(g)}{adg\phi(abh)} \\ \cdot \int_{(\frac{100}{\log Q})} \int_{(\frac{100}{\log Q})} (|L^4(1/2 + s_1, \chi)L^4(1/2 + s_2, \bar{\chi})| + \tau_4^2(g)) \left| \widetilde{\mathcal{W}}_2^\pm \left(s_1, s_2; \frac{dQ}{ghQ_0^2} \right) \right| ds_1 ds_2.$$

Notice first that Lemma 6.3 implies that, for any $k \geq 1$,

$$(51) \quad \sum_{\substack{g \\ b|g}} \frac{\tau^3(g)\tau_4^3(g)}{g} \sum_{d \leq D} \frac{\tau^3(d)}{d} \left| \widetilde{\mathcal{W}}_2^\pm \left(s_1, s_2, \frac{dQ}{ghQ_0^2} \right) \right| \\ \ll \frac{\left(1 + \frac{QD}{bhQ_0^2}\right)^{k-1} (\log Q)^2}{\max\{|s_1|, |s_2|\}^k |s_1 + s_2|^3} (\log D)^8 \sum_{\substack{g \\ b|g}} \frac{\tau^3(g)\tau_4^3(g)}{g} \exp\left(-c' \left(\frac{ghQ_0^2}{QD}\right)^{1/4}\right) \\ \ll \frac{\left(1 + \frac{QD}{bhQ_0^2}\right)^{k-1} (\log Q)^{O_k(1)}}{\max\{|s_1| + 1, |s_2| + 1\}^k (|s_1 + s_2| + 1)^3} \cdot \frac{\tau^3(b)\tau_4^3(b)}{b} \cdot \exp\left(-c' \left(\frac{bhQ_0^2}{QD}\right)^{1/4}\right)$$

Let us write $\ell = abh$ in (50) and split into dyadic blocks $\ell \sim L$ and $\max\{|s_1| + 1, |s_2| + 1\} \sim T$. Note that $\phi(abh) \gg abh/\log \log(abh)$ and $bh \geq L/A$. Using also the inequality $xy \leq x^2 + y^2$, the contribution of such a dyadic block to (50) is

$$(52) \quad \ll Q(\log Q)^{O(1)} \min_{k \in \{1, 4\}} \frac{\left(1 + \frac{ADQ}{LQ_0^2}\right)^{k-1} \log^\varepsilon(L)}{LT^k} \exp\left(-c' \left(\frac{LQ_0^2}{ADQ}\right)^{1/4}\right) \\ \cdot \sum_{\ell \sim L} \left(\sum_{\substack{abh=\ell \\ a \leq A \\ b, h > 0}} \frac{\tau^3(b)\tau_4^3(b)}{ab} \right) \sum_{\substack{\chi \pmod{\ell} \\ \chi \neq \chi_0}} \int_{-T}^T \left(1 + \left|L^8\left(\frac{1}{2} + \frac{100}{\log Q} + it, \chi\right)\right|\right) dt,$$

where the second integral in (50) is bounded by 1 due to the factor $\frac{1}{(|s_1 + s_2| + 1)^3}$. Here it suffices to make a rough estimate

$$(53) \quad \sum_{\substack{abh=\ell \\ a \leq A \\ b, h > 0}} \frac{\tau^3(b)\tau_4^3(b)}{ab} \ll \log A \prod_{p|\ell} \left(1 + \frac{2^9}{p}\right) \ll \log A \cdot \log L,$$

so that, by Proposition 3.2, the second line of (52) is at most

$$(54) \quad L^2 T^2 (\log(10LQT))^{O(1)}.$$

By (51), (52), and (54) above, (50) is bounded by

$$Q(\log Q)^{O(1)} \sum_{\substack{L, T \geq 1 \\ L=2^u, T=2^v}} \min_{k \in \{1, 4\}} L \frac{\left(1 + \frac{ADQ}{LQ_0^2}\right)^{k-1} \log^{O(1)}(LT)}{T^{k-2}} \exp\left(-c' \left(\frac{LQ_0^2}{ADQ}\right)^{1/4}\right).$$

Taking $k = 1$ when $T \leq 1 + \frac{ADQ}{LQ_0^2}$ and $k = 4$ otherwise, we see that this is

$$\begin{aligned} &\ll Q \sum_{\substack{L \geq 1 \\ L=2^u}} L \left(1 + \frac{ADQ}{LQ_0^2}\right) \log^{O(1)}(LQ) \exp\left(-c' \left(\frac{LQ_0^2}{ADQ}\right)^{1/4}\right) \\ &\ll \frac{ADQ^2 (\log Q)^{O(1)}}{Q_0^2} \ll_C \frac{Q^2}{(\log Q)^C} \end{aligned}$$

for any $C \geq 1$. Hence the claim follows. \square

Remark 6.5. In [1] the conditional bound (8) was used instead of Proposition 3.2. Since (8) does not require averaging over the modulus, in [1] it was possible to utilize averaging over ℓ when bounding the left hand side of (53). Due to this, the restriction $a \leq A$ (which is in place to make (53) hold) was not needed in the definition of $\mathcal{G}(\Psi, Q)$ in [1]. In our case, we needed to separate the terms with $a > A$ and treat them by a variant of the treatment of the case $d > D$ in [1].

7. EVALUATING $\mathcal{MS}(\Psi, Q) + \mathcal{MG}(\Psi, Q)$

We recall that $\mathcal{MS}(\Psi, Q)$ and $\mathcal{MG}(\Psi, Q)$ are defined in (27) and (32), respectively. We evaluate $\mathcal{MS}(\Psi, Q) + \mathcal{MG}(\Psi, Q)$ following [1, Section 8] but the details are somewhat different since we have the restriction $a \leq A$ in our sums.

Similarly to [1, Section 8] we use the Mellin transform of $\widetilde{\mathcal{W}}_1^\pm$ (from Lemma 6.1) to write $\mathcal{MG}(\Psi, Q)$ in terms of a contour integral with $\operatorname{Re} z = -\varepsilon < 0$ and shift the contour to $\operatorname{Re}(z) = \varepsilon > 0$. Here we pick up poles at $z = 0$ whose residue essentially cancels with $\mathcal{MS}(\Psi, Q)$. This process is recorded in the following lemma.

Lemma 7.1. *Let $C \geq 1$. Let $\mathcal{MS}(\Psi, Q)$ and $\mathcal{MG}(\Psi, Q)$ be as in (27) and (32) with parameters as in (49). Once Δ_0 is large enough in terms of C ,*

one has

(55)

$$\begin{aligned} \mathcal{MS}(\Phi, Q) + \mathcal{MG}(\Phi, Q) &= \frac{Q}{2} \sum_{\substack{m, n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \frac{1}{2\pi i} \int_{(\varepsilon)} \widetilde{\mathcal{W}}_1 \left(\frac{mQ_0^2}{Q^2}, \frac{nQ_0^2}{Q^2}; z \right) \\ &\quad \cdot \frac{\zeta(1-z)\mathcal{F}(-z, g, MN)}{\zeta(1+z)\phi(gMN, 1+z)} \left(\frac{Q}{gQ_0^2} \right)^{-z} dz + O \left(\frac{Q^2}{(\log Q)^C} \right), \end{aligned}$$

where

$$\phi(r, s) = \prod_{p|r} \left(1 - \frac{1}{p^s} \right),$$

and

$$\begin{aligned} \mathcal{F}(s, g, MN) &= \phi(MN, s+1) \prod_{p|gMN} \left(1 - \frac{1}{p(p-1)} + \frac{1}{p^{1+s}(p-1)} \right) \\ &\quad \times \prod_{\substack{p|g \\ p \nmid MN}} \left(1 - \frac{1}{p^{1+s}} - \frac{1}{p-1} \left(1 - \frac{1}{p^s} \right) \right). \end{aligned}$$

Proof. Write

$$S(a, b, d, g, M, N) := \sum_{\pm} \sum_{\substack{h>0 \\ (h, MN)=1}} \frac{1}{\phi(abh)} \mathcal{W}^{\pm} \left(\frac{gMQ_0^2}{Q^2}, \frac{gNQ_0^2}{Q^2}; \frac{dQ}{ghQ_0^2} \right),$$

so that

$$\mathcal{MG}(\Psi, Q) = \frac{Q}{2} \sum_{\substack{m, n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{d \leq D \\ (d, gMN)=1}} \sum_{\substack{a \leq A \\ (a, gMN)=1}} \sum_{\substack{b|g \\ (b, MN)=1}} \frac{\mu(a)\mu(b)\mu(d)}{ad} S(a, b, d, g, M, N).$$

where as usual $g := (m, n)$, $M := m/g$ and $N := n/g$.

Using the Mellin transform of $\widetilde{\mathcal{W}}_1^{\pm}$ given in Lemma 6.1 with $c = -\varepsilon < 0$, we obtain that

(56)

$$S(a, b, d, g, M, N) = \sum_{\pm} \sum_{\substack{h=1 \\ (h, MN)=1}}^{\infty} \frac{1}{\phi(abh)} \frac{1}{2\pi i} \int_{(-\varepsilon)} \widetilde{\mathcal{W}}_1^{\pm} \left(\frac{gMQ_0^2}{Q^2}, \frac{gNQ_0^2}{Q^2}; z \right) \left(\frac{dQ}{ghQ_0^2} \right)^{-z} dz.$$

We can interchange the sum and the integral since the sum over h is absolutely convergent for $\operatorname{Re}(z) < 0$. Writing out the Euler product, we obtain that, for $(ab, MN) = 1$ and $\operatorname{Re}(s) > 0$, one has

(57)

$$\sum_{\substack{h=1 \\ (h, MN)=1}}^{\infty} \frac{1}{\phi(abh)h^s} = \frac{1}{\phi(ab)} \zeta(s+1) \prod_{p|MN} \left(1 - \frac{1}{p^{s+1}} \right) \prod_{p \nmid abMN} \left(1 + \frac{1}{p^s} \cdot \frac{1}{p(p-1)} \right).$$

Therefore

$$S(a, b, d, g, M, N) = \sum_{\pm} \frac{1}{2\pi i} \int_{(-\varepsilon)} \widetilde{\mathcal{W}}_1^{\pm} \left(\frac{gMQ_0^2}{Q^2}, \frac{gNQ_0^2}{Q^2}; z \right) \zeta(1-z) \left(\frac{dQ}{gQ_0^2} \right)^{-z} \\ \cdot \frac{1}{\phi(ab)} \prod_{p|MN} \left(1 - \frac{1}{p^{1-z}} \right) \prod_{p \nmid abMN} \left(1 + \frac{p^z}{p(p-1)} \right) dz.$$

Next we move the integration line to $\operatorname{Re} z = \varepsilon$. We encounter a pole at $z = 0$, leading to a main term $-(\text{Residue at } z = 0)$ that equals

$$\widetilde{\mathcal{W}}_1^{\pm} \left(\frac{gMQ_0^2}{Q^2}, \frac{gNQ_0^2}{Q^2}; 0 \right) \frac{1}{\phi(ab)} \frac{\phi(MN)}{MN} \prod_{p \nmid abMN} \left(1 + \frac{1}{p(p-1)} \right).$$

Here, by first substituting $u' = u|x \pm y|$ in the definition of $\widetilde{\mathcal{W}}_1^{\pm}$ and then using (31),

$$\widetilde{\mathcal{W}}_1^{\pm} \left(\frac{gMQ_0^2}{Q^2}, \frac{gNQ_0^2}{Q^2}; 0 \right) = \int_0^{\infty} \Psi(u) V \left(\frac{gMQ_0^2}{Q^2}, \frac{gNQ_0^2}{Q^2}, u \right) du \\ = \int_0^{\infty} \Psi(u) V \left(gM, gN, \frac{uQ}{Q_0} \right) du.$$

Therefore, writing $\mathcal{MG}_1(\Psi, Q)$ for the contribution of the residue term at $z = 0$ we obtain

$$(58) \quad \mathcal{MG}(\Psi, Q) = \mathcal{MG}_1(\Psi, Q) + \mathcal{MG}_2(\Psi, Q),$$

where

$$(59) \quad \mathcal{MG}_1(\Psi, Q) := Q \sum_{\substack{m, n=1 \\ m \neq n}} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \frac{\phi(MN)}{MN} \sum_{\substack{d \leq D \\ (d, mn)=1}} \frac{\mu(d)}{d} \sum_{\substack{a \leq A \\ (a, mn)=1}} \sum_{\substack{b|g \\ (b, MN)=1}} \frac{\mu(a)\mu(b)}{a\phi(ab)} \\ \cdot \prod_{p \nmid abMN} \left(1 + \frac{1}{p(p-1)} \right) \int_0^{\infty} \Psi(u) V \left(gM, gN, \frac{uQ}{Q_0} \right) du,$$

where we have changed the factor of $Q/2$ to Q due to summing over \pm . Moreover,

$$(60) \quad \mathcal{MG}_2(\Psi, Q) := \frac{Q}{2} \sum_{\substack{m, n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \frac{1}{2\pi i} \int_{(\varepsilon)} \sum_{\substack{d \leq D \\ (d, gMN)=1}} \sum_{\substack{a \leq A \\ (a, mn)=1}} \sum_{\substack{b|g \\ (b, MN)=1}} \frac{\mu(a)\mu(b)\mu(d)}{a\phi(ab)d^{1+z}} \\ \cdot \widetilde{\mathcal{W}}_1 \left(\frac{gMQ_0^2}{Q^2}, \frac{gNQ_0^2}{Q^2}; z \right) \zeta(1-z) \left(\frac{Q}{gQ_0^2} \right)^{-z} \prod_{p|MN} \left(1 - \frac{1}{p^{1-z}} \right) \prod_{p \nmid abMN} \left(1 + \frac{p^z}{p(p-1)} \right) dz,$$

where we recall that $\widetilde{\mathcal{W}}_1^+ + \widetilde{\mathcal{W}}_1^- = \widetilde{\mathcal{W}}_1$. We will first show that

$$(61) \quad \mathcal{MG}_1(\Psi, Q) + \mathcal{MS}(\Psi, Q) = O\left(\frac{DQ^2(\log Q)^{O(1)}}{Q_0}\right).$$

By definition (27),

$$\mathcal{MS}(\Psi, Q) = - \sum_{\substack{m, n=1 \\ m \neq n}} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{a \leq A, d \leq D \\ (ad, mn)=1}} \sum_{\substack{l > 0 \\ (l, MN)=1}} \mathbf{1}_{(l, g)=1} \Psi\left(\frac{adl}{Q}\right) \frac{\mu(a)\mu(d)l}{\phi(ad)} V\left(m, n; \frac{adl}{Q_0}\right).$$

Writing $\mathbf{1}_{(l, g)=1} = \sum_{l=bk, b|g} \mu(b)$, we see that

$$\mathcal{MS}(\Psi, Q) = - \sum_{\substack{m, n=1 \\ m \neq n}} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{a \leq A, d \leq D \\ (ad, mn)=1}} \sum_{\substack{b|g, k > 0 \\ (bk, MN)=1}} \Psi\left(\frac{abdk}{Q}\right) \frac{\mu(a)\mu(b)\mu(d)bk}{\phi(abk)} V\left(m, n; \frac{abdk}{Q_0}\right).$$

Writing $\mathcal{U}(m, n, u) = V(m, n, u/Q_0)\Psi(u/Q)$ and using Mellin inversion, we see that

$$\begin{aligned} \mathcal{MS}(\Psi, Q) &= - \sum_{\substack{m, n=1 \\ m \neq n}} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{a \leq A, d \leq D \\ (ad, mn)=1}} \sum_{\substack{b|g \\ (b, MN)=1}} \sum_{\substack{k > 0 \\ (k, MN)=1}} \frac{\mu(a)\mu(b)\mu(d)bk}{\phi(abk)} \\ &\quad \times \frac{1}{2\pi i} \int_{(1+\varepsilon)} \frac{\widetilde{\mathcal{U}}(m, n, z)}{(abdk)^z} dz, \end{aligned}$$

where

$$\widetilde{\mathcal{U}}(m, n, z) = \int_0^\infty \mathcal{U}(m, n, u) u^z \frac{du}{u}.$$

Noting that the sum over k above is absolutely convergent for $\operatorname{Re} z > 1$, we can interchange the order of summation and integration. Recalling (57) we see that

$$\begin{aligned} \mathcal{MS}(\Psi, Q) &= - \sum_{\substack{m, n=1 \\ m \neq n}} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \sum_{\substack{b|g \\ (b, MN)=1}} \frac{1}{2\pi i} \int_{(1+\varepsilon)} \sum_{\substack{a \leq A, d \leq D \\ (ad, mn)=1}} \frac{\mu(a)\mu(b)\mu(d)b}{\phi(ab)(abdk)^z} \zeta(z) \\ &\quad \cdot \prod_{p|MN} \left(1 - \frac{1}{p^z}\right) \prod_{p \nmid abMN} \left(1 + \frac{1}{p^{z-1}} \cdot \frac{1}{p(p-1)}\right) \widetilde{\mathcal{U}}(m, n, z) dz. \end{aligned}$$

Next we move the integration to the line $\operatorname{Re} z = 1/\log Q$. Since

$$\widetilde{\mathcal{U}}(m, n, 1) = Q \int_0^\infty \Psi(u) V\left(m, n; \frac{uQ}{Q_0}\right) du,$$

the residue from the pole at $z = 1$ equals $-\mathcal{MG}_1(\Psi, Q)$ and the remaining integral can be included in the error term in (61) since $\widetilde{\mathcal{U}}(m, n; z)$ decays rapidly when m or n is $\gg Q^2/Q_0$ (due to Lemma 4.2) or $|\operatorname{Im} z|$ grows.

Therefore (61) holds and the remaining main term of $\mathcal{MG}(\Psi, Q) + \mathcal{MS}(\Psi, Q)$ is $\mathcal{MG}_2(\Psi, Q)$ defined by (60). We shift the contour in (60) to $\operatorname{Re}(z) = 1 - \frac{1}{\log Q}$. The sums over a and d can be extended to all positive integers

with an error $\ll Q^2/(\log Q)^C$ using Lemma 6.1 when Δ_0 is large enough depending on C . For the d -sum, this was done in [1, Proof of Lemma 8.1] and one can argue similarly for the a -sum. Hence, apart from an acceptable error, $\mathcal{MG}_2(\Psi, Q)$ equals

$$\begin{aligned} & \frac{Q}{2} \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \frac{1}{2\pi i} \int_{(1-1/\log Q)} \widetilde{\mathcal{W}}_1 \left(\frac{gMQ_0^2}{Q^2}, \frac{gNQ_0^2}{Q^2}; z \right) \zeta(1-z) \left(\frac{Q}{gQ_0^2} \right)^{-z} \\ & \cdot \sum_{\substack{d>0 \\ (d,gMN)=1}} \sum_{\substack{a>0 \\ (a,mn)=1}} \sum_{\substack{b|g \\ (b,MN)=1}} \frac{\mu(a)\mu(b)\mu(d)}{a\phi(ab)d^{1+z}} \prod_{p|MN} \left(1 - \frac{1}{p^{1-z}} \right) \prod_{p|abMN} \left(1 + \frac{p^z}{p(p-1)} \right) dz. \end{aligned}$$

Writing $r = ab$, the second line equals

$$= \sum_{\substack{d>0 \\ (d,gMN)=1}} \sum_{\substack{r>0 \\ (r,MN)=1}} \frac{\mu(d)\mu(r)(r,g)}{r\phi(r)d^{1+z}} \prod_{p|MN} \left(1 - \frac{1}{p^{1-z}} \right) \prod_{p|rMN} \left(1 + \frac{p^z}{p(p-1)} \right).$$

Careful calculation with the Euler products reveals that this equals

$$\frac{\mathcal{F}(-z, g, MN)}{\zeta(1+z)\phi(gMN, 1+z)}$$

and thus

$$\begin{aligned} \mathcal{MG}_2(\Psi, Q) &= \frac{Q}{2} \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \frac{1}{2\pi i} \int_{(1-1/\log Q)} \frac{\zeta(1-z)\mathcal{F}(-z, g, MN)}{\zeta(1+z)\phi(gMN, 1+z)} \\ & \cdot \widetilde{\mathcal{W}}_1 \left(\frac{gMQ_0^2}{Q^2}, \frac{gNQ_0^2}{Q^2}; z \right) \left(\frac{Q}{gQ_0^2} \right)^{-z} dz + O \left(\frac{Q^2}{(\log Q)^C} \right). \end{aligned}$$

Now the Lemma follows from moving the contour back to $\operatorname{Re} z = \varepsilon$ along with (58) and (61). \square

Next we will prove the following proposition which evaluates the main term in Lemma 7.1. In [1] this proposition was conditional on the Lindelöf hypothesis, but here we prove it unconditionally.

Proposition 7.2. *Let $\mathcal{MS}(\Psi, Q)$ and $\mathcal{MG}(\Psi, Q)$ be as in (27) and (32) with parameters as in (49). Once Δ_0 is large enough, one has*

$$\begin{aligned} \mathcal{MS}(\Psi, Q) + \mathcal{MG}(\Psi, Q) &= \frac{-53524}{16!} Q^2 (\log Q)^{16} \frac{\widetilde{\Psi}(2) \mathcal{K} \left(\frac{1}{2}, \frac{1}{2}; 1 \right)}{2 \zeta(2)} \int_{-\infty}^{\infty} G \left(\frac{1}{2}, t \right) dt \\ & \quad + O(Q^2 (\log Q)^{15}). \end{aligned}$$

with $\mathcal{K}(s_1, s_2; z)$ implicitly defined below Equation (38) in [1].

Proof. First we apply (37) from Lemma 6.2 to (55) to obtain that

(62)

$$\begin{aligned}
\mathcal{MS}(\Psi, Q) + \mathcal{MG}(\Psi, Q) &= \frac{Q}{2} \sum_{\substack{m, n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{\sqrt{mn}} \frac{1}{(2\pi i)^3} \int_{(\varepsilon)} \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} \widetilde{\mathcal{W}}_3(s_1, s_2; z) \\
&\cdot \frac{\zeta(1-z)\mathcal{F}(-z, g, MN)}{\zeta(1+z)\phi(gMN, 1+z)} \left(\frac{Q}{gQ_0^2}\right)^{-z} \left(\frac{Q^2}{mQ_0^2}\right)^{s_1} \left(\frac{Q^2}{nQ_0^2}\right)^{s_2} ds_2 ds_1 dz \\
&+ O\left(\frac{Q^{3+3\varepsilon}}{T^{1-\varepsilon}} \sum_{\substack{m, n=1 \\ m \neq n}}^{\infty} \frac{\tau_4(m)\tau_4(n)}{m^{1+\varepsilon}n^{1+\varepsilon}} \frac{1}{|\log(\frac{m}{n})|}\right) + O(Q^2),
\end{aligned}$$

where the integration over z gives a factor $\ll 1$ in the error term due to the rapid decay in z in (37). We choose the height $T := Q^{5/4}$. Let us first consider the error term. We divide the sum over m and n into two ranges. The first case is when $|m - n| > \frac{1}{4}n$. In this case, $|\log(\frac{m}{n})| \gg 1$ and thus the contribution to the error term in (62) is

$$\ll Q^{\frac{7}{4}+5\varepsilon} \sum_{m, n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{(mn)^{1+\varepsilon}} \ll Q^{\frac{7}{4}+5\varepsilon}.$$

For the other range $|m - n| \leq \frac{1}{4}n$ we have

$$\left|\log\left(\frac{m}{n}\right)\right| \gg \frac{|m - n|}{n}.$$

Thus the contribution from this range to the error term in (62) is

$$\ll Q^{\frac{7}{4}+5\varepsilon} \sum_{n=1}^{\infty} \frac{\tau_4(n)}{n^{1+\varepsilon}} \sum_{\substack{m \neq n \\ \frac{3}{4}n \leq m \leq \frac{5}{4}n}} \frac{\tau_4(m)}{m^{1+\varepsilon}} \frac{n}{|m - n|} \ll Q^{\frac{7}{4}+5\varepsilon} \sum_{n=1}^{\infty} \frac{\tau_4(n)}{n^{1+\varepsilon}} \sum_{j=1}^n \frac{1}{j} \ll Q^{\frac{7}{4}+5\varepsilon}.$$

Before we move the contour integral, we reinsert the terms $m = n$ to the main term of (62). The contribution of this addition is

$$\begin{aligned}
&\frac{Q}{2} \sum_{n=1}^{\infty} \frac{\tau_4^2(n)}{n} \frac{1}{(2\pi i)^3} \int_{(\varepsilon)} \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} \widetilde{\mathcal{W}}_3(s_1, s_2; z) \\
&\cdot \frac{\zeta(1-z)\mathcal{F}(-z, n, 1)}{\zeta(1+z)\phi(n, 1+z)} \left(\frac{Q}{nQ_0^2}\right)^{-z} \left(\frac{Q^2}{nQ_0^2}\right)^{s_1+s_2} ds_2 ds_1 dz.
\end{aligned}$$

Let us now show that this contribution is acceptable. We can move the contour integral over s_1 and s_2 to $\text{Re}(s_i) = \varepsilon$. We encounter no poles, and the sum over n is absolutely convergent. The resulting vertical integral is bounded by $Q^{1+3\varepsilon}$, and by (38), the contribution from horizontal integrals is bounded by

$$\frac{Q^{3+4\varepsilon}}{T^{1-\varepsilon}} \ll Q^{\frac{7}{4}+6\varepsilon}.$$

Hence, apart from an acceptable error, (62) equals

(63)

$$\frac{Q}{2} \frac{1}{(2\pi i)^3} \int_{(\varepsilon)} \int_{1/2+\varepsilon-iT}^{1/2+\varepsilon+iT} \int_{1/2+\varepsilon-iT}^{1/2+\varepsilon+iT} \widetilde{\mathcal{W}}_3(s_1, s_2; z) \frac{\zeta(1-z)}{\zeta(1+z)} \frac{Q^{2s_1+2s_2-z}}{Q_0^{2(s_1+s_2-z)}} \mathcal{J}(s_1, s_2; z) ds_2 ds_1 dz,$$

where

$$\mathcal{J}(s_1, s_2; z) = \sum_{m,n=1}^{\infty} \frac{\tau_4(m)\tau_4(n)}{m^{1/2+s_1}n^{1/2+s_2}} \frac{g^z \mathcal{F}(-z, g, MN)}{\phi(gMN, 1+z)}.$$

As in [1, Proof of Proposition 9.1], writing out the Euler product, one can see that

$$\mathcal{J}(s_1, s_2; z) = \zeta(2-z) \frac{\zeta^4\left(\frac{1}{2}+s_1\right)\zeta^4\left(\frac{1}{2}+s_2\right)}{\zeta^4\left(\frac{3}{2}+s_1-z\right)\zeta^4\left(\frac{3}{2}+s_2-z\right)} \zeta^{16}(1+s_1+s_2-z) \mathcal{K}(s_1, s_2; z),$$

where $\mathcal{K}(s_1, s_2; z) = \prod_p \mathcal{K}_p(s_1, s_2; z)$ is absolutely convergent when

$$\operatorname{Re}(s_1) > 0, \operatorname{Re}(s_2) > 0, \operatorname{Re}(s_1 + s_2) > \operatorname{Re}(z) - 1/2$$

and

$$\operatorname{Re}(z) \in (0, 3/2), \text{ and } \operatorname{Re} z < 1 + \operatorname{Re} s_i.$$

Now we move the lines of integration in (63) to $\operatorname{Re}(s_1) = \operatorname{Re}(s_2) = 2\varepsilon$, encountering poles of order four at $s_1 = 1/2$ and $s_2 = 1/2$. By the Weyl bound (see e.g. [12, Theorem 5.12]) and the Phragmen-Lindelöf principle one has, for $\sigma \geq 0$,

$$|\zeta(1/2 + \sigma + it)| \ll T^{\max\{0, 1/6 - \sigma/3\} + \varepsilon},$$

so that, using also (38), the horizontal integrals contribute

$$\begin{aligned} & Q^{1+\varepsilon} \max_{2\varepsilon \leq \sigma \leq 1/2+\varepsilon} \frac{T^{8 \max\{0, 1/6 - \sigma/3\} + 8\varepsilon} Q^{4\sigma}}{T^{1-\varepsilon}} \\ & \leq Q^{1+\varepsilon} \frac{Q^{4(1/2+\varepsilon)} T^{9\varepsilon}}{T} + Q^{1+\varepsilon} T^{9\varepsilon} \max_{2\varepsilon \leq \sigma \leq 1/2} T^{1/3} \left(\frac{Q}{T^{2/3}}\right)^{4\sigma} \\ & = O(Q^{7/4+20\varepsilon}). \end{aligned}$$

Furthermore, using again (38), the integrals over $\operatorname{Re} s_i = 2\varepsilon$ contribute to (63)

$$\ll \max_{1 \leq T' \leq T} Q^{1+\varepsilon} \frac{1}{T'^{1-\varepsilon}} \int_{-T'}^{T'} |\zeta(1/2 + 2/\log Q + it)|^8 dt.$$

By Lemma 3.5 this is

$$\ll Q^{1+\varepsilon} \max_{1 \leq T' \leq T} \frac{1}{T'^{1-\varepsilon}} T'^{3/2+\varepsilon} \ll Q^{1+4\varepsilon} T^{1/2} \ll Q^{7/4}.$$

Finally by the definition of $\widetilde{\mathcal{W}}_3(s_1, s_2; z)$ in (34), the main contribution of the residues is

$$\frac{Q}{2} \frac{1}{2\pi i} \int_{(\epsilon)} \frac{\widetilde{\Psi}(3-z)}{\pi^{2-2z}} \left(\int_{-\infty}^{\infty} \mathcal{H}\left(\frac{z}{2} - it, z\right) G\left(\frac{1}{2} + \frac{1-z}{2}, t\right) dt \right) \frac{\zeta(1-z)}{\zeta(1+z)} \zeta(2-z) \mathcal{K}\left(\frac{1}{2}, \frac{1}{2}; z\right) \\ \times \left(\operatorname{Res}_{s_1=s_2=1/2} \frac{\zeta^4\left(\frac{1}{2} + s_1\right) \zeta^4\left(\frac{1}{2} + s_2\right) \zeta^{16}(1 + s_1 + s_2 - z)}{(s_1 + s_2 - z) \zeta^4\left(\frac{3}{2} + s_1 - z\right) \zeta^4\left(\frac{3}{2} + s_2 - z\right)} \frac{Q^{2s_1+2s_2-z}}{Q_0^{2(s_1+s_2-z)}} \right) dz.$$

Then we follow the residue calculation of Proposition 9.1 in [1] and obtain the main term. □

Now we have handled unconditionally all the terms involved and Theorem 1.1 follows as in [1, Section 10].

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MATHEMATICS DEPARTMENT, KANSAS STATE UNIVERSITY, 138 CARDWELL HALL MANHATTAN, KS 66506 USA

Email address: `chandee@ksu.edu`

MATHEMATICS DEPARTMENT, KANSAS STATE UNIVERSITY, 138 CARDWELL HALL MANHATTAN, KS 66506 USA

Email address: `xiannan@ksu.edu`

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF TURKU, 20014 TURKU, FINLAND

Email address: `ksmato@utu.fi`

DEPARTMENT OF MATHEMATICS, LUNT HALL, ROOM 201, NORTHWESTERN UNIVERSITY, 2033 SHERIDAN ROAD, EVANSTON, IL 60208, USA

Email address: `maksym.radziwill@gmail.com`