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Research Article

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Pseudometric spaces: From minimality to maximality in the groups of combinatorial self-similarities

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Abstract: The group of combinatorial self-similarities of a pseudometric space (X, d) is the maximal subgroup of the symmetric group $\mathrm{Sym}(X)$ whose elements preserve the four-point equality $d(x, y) = d(u, v)$. Let us denote by $\mathcal{IP}$ the class of all pseudometric spaces (X, d) for which every combinatorial self-similarity $\Phi : X \to X$ satisfies the equality $d(x, \Phi(x)) = 0$, but all permutations of metric reflection of (X, d) are combinatorial self-similarities of this reflection. The structure of $\mathcal{IP}$-spaces is fully described.

Keywords: combinatorial similarity; discrete pseudometric; equivalence relation; strongly rigid pseudometric; symmetric group

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1 Introduction

The present article is aimed at describing some interconnections between pseudometric spaces, combinatorial similarities, and equivalence relations [1,2,5–7].

The concept of combinatorial similarity for pseudometrics and, more generally, for mappings defined on the Cartesian square of a set, was introduced by Dovgoshey and Luukkainen [7]. Theorem 3.1 in [7] contains a necessary and sufficient condition under which a mapping is combinatorially similar to a pseudometric. The strongly rigid pseudometrics and discrete ones were characterized, up to combinatorial similarity, in Theorems 4.13 and 3.9 of [7], respectively.

The main result of the study by Bilet and Dovgoshey [2], Theorem 2.8, completely describes the structure of semimetric spaces (X, d) for which the group of combinatorial self-similarities of (X, d) coincides with the symmetric group $\mathrm{Sym}(X)$. In particular, Theorem 2.8 shows that every permutation of X is a combinatorial self-similarity of (X, d) if d is discrete or strongly rigid.

The aforementioned results are the starting points of our research. To describe the general direction of this research, we note that the transition from the pseudometric space (X, d) to its metric reflection $(X/\overset{0}{=}, \delta_d)$ naturally generates a homomorphism $H : \mathbf{Cs}(X, d) \to \mathbf{Cs}(X/\overset{0}{=}, \delta_d)$ of the groups of combinatorial self-similarities of (X, d) and $(X/\overset{0}{=}, \delta_d)$. How are the algebraic properties of the homomorphism H related to the metric properties of (X, d)?

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In the present article, we give a metric characterization of pseudometric spaces (X, d) for which the kernel of H coincides with $\mathbf{Cs}(X, d)$, but the equality

$$\mathbf{Cs}(X/\overset{0}{=}, \delta_d) = \operatorname{Sym}(X/\overset{0}{=})$$

holds.

2 Preliminaries

Let us start from the classical notion of metric space.

A *metric* on a set X is a function $d : X^2 \to \mathbb{R}$ such that for all x, $y, z \in X$:

(i) $d(x, y) \geqslant 0$ with equality if and only if $x = y$, the *positivity property*;
(ii) $d(x, y) = d(y, x)$, the *symmetry property*;
(iii) $d(x, y) \leqslant d(x, z) + d(z, y)$, the *triangle inequality*.

In 1934, Kurepa [15] introduced the pseudometric spaces which, unlike metric spaces, allow the zero distance between different points.

Definition 2.1. Let X be a set, and let $d : X^2 \to \mathbb{R}$ be a nonnegative, symmetric function such that $d(x, x) = 0$ for every $x \in X$. The function d is a *pseudometric* on X if it satisfies the triangle inequality.

If d is a pseudometric on X, we say that (X, d) is a *pseudometric space*. We will denote by $d(X^2)$ the range of the pseudometric d,

$$d(X^2) \coloneqq \{d(x, y) : x, y \in X\}.$$

Definition 2.2. [7] Let (X, d) and (Y, ρ) be pseudometric spaces. The spaces (X, d) and (Y, ρ) are *combinatorially similar* if there exist bijections $\Psi : Y \to X$ and $f : d(X^2) \to \rho(Y^2)$ such that

$$\rho(x, y) = f\big(d(\Psi(x), \Psi(y))\big) \tag{2.1}$$

for all x, $y \in Y$. In this case, we will say that Ψ is a *combinatorial similarity* of (Y, ρ) and (X, d).

Let (X, d) and (Y, ρ) be metric spaces. A mapping $\Psi : Y \to X$ is a *similarity* if Ψ is bijective, and there is $r > 0$, the *ratio* of Ψ, such that

$$d(\Psi(x), \Psi(y)) = r\rho(x, y)$$

for all x, $y \in Y$. It is clear that every similarity is a combinatorial similarity. Furthermore, a combinatorial similarity $\Psi : Y \to X$ is a similarity with a ratio r if (1) holds for all x, $y \in Y$ with $f(t) = r^{-1}t$ for every $t \in d(X^2)$.

The following simple example illustrates the difference between similarity and combinatorial similarity.

Example 2.3. Let A and B be triangles on the Euclidean plane. Then A and B are combinatorially similar iff both these triangles are simultaneously equilateral, simultaneously isosceles, or simultaneously scalene.

The basic morphisms of the theory of metric spaces, the isometries of these spaces, can defined as similarities with the ratio 1. Thus, the notion of combinatorial similarities can be considered as a generalization of the notion of the isometries of metric spaces.

Remark 2.4. The notion of isometry of metric spaces can be extended to pseudometric spaces in various nonequivalent ways. For example, Kelley [13] defines the isometries of pseudometric spaces (X, d) and (Y, ρ) as the distance-preserving surjections $X \to Y$.

In particular, we also will use the following generalization of isometrics.

Definition 2.5. [1] Let (X, d) and (Y, ρ) be pseudometric spaces. A mapping $\Phi : X \to Y$ is a *pseudoisometry* of (X, d) and (Y, ρ) if:

(i) $\rho(\Phi(x), \Phi(y)) = d(x, y)$ holds for all $x, y \in X$;

(ii) For every $u \in Y$ there is $v \in X$ such that $\rho(\Phi(v), u) = 0$.

We say that two pseudometric spaces are *pseudoisometric* if there is a pseudoisometry of these spaces.

For every pseudometric spaces (X, d), the set of all combinatorial self-similarities is a group with the function composition as a group operation. The identity mapping,

$$\mathrm{Id}_X : X \to X, \quad \mathrm{Id}_X(x) = x \text{ for every } x \in X,$$

is the identity element of this group. If $d : X^2 \to \mathbb{R}$ is a metric, then we can characterize Id_X as a unique mapping $X \xrightarrow{f} X$, which satisfies the equality

$$d(x, f(x)) = 0 \tag{2.2}$$

for every $x \in X$. For the case when the pseudometric $d : X^2 \to \mathbb{R}$ is not a metric, one can always find a bijection $X \xrightarrow{f} X$ such that $f \neq \mathrm{Id}_X$ but (2.2) holds for every $x \in X$.

Example 2.6. Let (X, d_0) be a pseudometric space endowed by the *zero pseudometric*, $d(x, y) = 0$ for all x, $y \in X$. Then (2.2) holds for every $x \in X$ and each $X \xrightarrow{f} X$.

Definition 2.7. Let (X, d) be a pseudometric space. A bijection $f : X \to X$ is a *pseudoidentity* if the equality

$$d(x, f(x)) = 0$$

holds for every $x \in X$.

Remark 2.8. It is clear that, for every pseudometric space (X, d), the set of all pseudoidentities $X \to X$ is a subgroup of the group of all combinatorial self-similarities of (X, d).

The combinatorial similarities of pseudometric spaces are the main type of morphisms studied in this article.

The groups of all combinatorial self-similarities and all pseudoidentities of a pseudometric space (X, d) will be denoted by $\mathbf{Cs}(X, d)$ and $\mathbf{PI}(X, d)$, respectively. Thus, for every pseudometric space (X, d), we have

$$\mathbf{PI}(X, d) \subseteq \mathbf{Cs}(X, d) \subseteq \mathrm{Sym}(X),$$

where $\mathrm{Sym}(X)$ is the symmetric group of all permutations of the set X.

For every nonempty pseudometric space (X, d), we define a binary relation $\overset{0(d)}{=}$ on X by

$$(x \overset{0(d)}{=} y) \Leftrightarrow (d(x, y) = 0), \tag{2.3}$$

for all $x, y \in X$.

In the future, we will simply write $\overset{0}{=}$ instead of $\overset{0(d)}{=}$, when it is clear, which d we are talking about.

The proof of the following proposition can be found in the study by Kelley [13, Chapter 4, Theorem 15].

Proposition 2.9. *Let X be a nonempty set, and let $d : X^2 \to \mathbb{R}$ be a pseudometric on X. Then $\overset{0}{=}$ is an equivalence relation on X and, in addition, the function δ_d,*

$$\delta_d(\alpha,\beta) \coloneqq d(x,y), \quad x \in \alpha \in X/\overset{0}{=}, \quad y \in \beta \in X/\overset{0}{=}, \tag{2.4}$$

is a correctly defined metric on the quotient set $X/\overset{0}{=}$.

In what follows, we will say that the metric space $(X/\overset{0}{=}, \delta_d)$ is the *metric reflection* of (X, d).

The following fact was proved in Theorem 3.3 of [1].

Lemma 2.10. *Pseudometric spaces (X, d) and (Y, ρ) are pseudoisometric if and only if the metric reflections $(X/\overset{0}{=}, \delta_d)$ and $(Y/\overset{0}{=}, \delta_\rho)$ are isometric metric spaces.*

Let us define a class $\mathcal{IP}$ of pseudometric spaces as follows.

Definition 2.11. A pseudometric space (X, d) belongs to $\mathcal{IP}$ if the equalities

$$\mathbf{Cs}(X,d) = \mathbf{PI}(X,d) \tag{2.5}$$

and

$$\mathbf{Cs}(X/\overset{0}{=}, \delta_d) = \operatorname{Sym}(X/\overset{0}{=}) \tag{2.6}$$

hold. We will say that (X, d) is a $\mathcal{IP}$-space if $(X, d) \in \mathcal{IP}$.

Thus, (X, d) is a $\mathcal{IP}$-space if and only if the group $\mathbf{Cs}(X, d)$ is as small as possible, but the group $\mathbf{Cs}(X/\overset{0}{=}, \delta_d)$ is as large as possible.

Example 2.12. Let (X, d) be a nonempty metric space. Then $(X, d) \in \mathcal{IP}$ holds if and only if $|X| = 1$. Indeed, the implication

$$(|X| = 1) \Rightarrow ((X,d) \in \mathcal{IP})$$

is evidently valid. Let (X, d) belong to $\mathcal{IP}$. To prove the equality $|X| = 1$, we note that $\mathbf{PI}(X, d)$ contains the mapping $\mathrm{Id}_X : X \to X$ only and that $\mathbf{Cs}(X, d)$ and $\mathbf{Cs}(X/\overset{0}{=}, \delta_d)$ are isomorphic groups because (X, d) is a metric space. Hence, (2.5) implies the equality $|\mathbf{Cs}(X/\overset{0}{=}, \delta_d)| = 1$. By using the last equality and (2.6), we obtain $|\operatorname{Sym}(X/\overset{0}{=})| = 1$, which is possible if and only if $|X/\overset{0}{=}| = 1$. Since d is a metric, we also have $|X| = |X/\overset{0}{=}|$. The equality $|X| = 1$ follows.

The main goal of this article is to describe the structure of $\mathcal{IP}$-spaces. To do this, we introduce into consideration pseudometric generalizations of some well known classes of metric spaces.

Let (X, d) be a metric space. Recall that the metric d is said to be *strongly rigid* if, for all $x, y, u, v \in X$, the condition

$$d(x,y) = d(u,v) \neq 0 \tag{2.7}$$

implies

$$(x = u \text{ and } y = v) \text{ or } (x = v \text{ and } y = u). \tag{2.8}$$

It should be noted that a strongly rigid metric on a set X exists if and only if $|X| \leqslant 2^{\aleph_0}$, since, for each infinite X and every strongly rigid $d : X^2 \to \mathbb{R}$, the sets X, X^2 and $d(X^2)$ have the same cardinality,

$$|X| = |X^2| = |d(X^2)|,$$

and $|d(X^2)| \leqslant |\mathbb{R}| = 2^{\aleph_0}$. Some other properties of strongly rigid metric spaces are described in previous studies [3,4,7–12,16,18].

The concept of strongly rigid metric can be naturally generalized to the concept of *strongly rigid pseudometric*, as was done in the study by Dovgoshey and Luukkainen [7].

Definition 2.13. Let (X, d) be a pseudometric space. The pseudometric d is *strongly rigid* if every metric subspace of (X, d) is strongly rigid.

Remark 2.14. A pseudometric $d : X^2 \to \mathbb{R}$ is *strongly rigid* if and only if (2.7) implies

$$(d(x,u) = d(y,v) = 0) \text{ or } (d(x,v) = d(y,u) = 0) \tag{2.9}$$

for all $x, y, u, v \in X$.

Example 2.15. The implication (2.7) ⇒ (2.9) is vacuously true for the zero pseudometric $d : X^2 \to \mathbb{R}$. Hence, the zero pseudometric is strongly rigid.

The well known example of pseudometric space is a seminormed vector space. Recall that a *seminorm* on a vector space X is a function $\|\cdot\| : X \to \mathbb{R}$ such that $\|x + y\| \leqslant \|x\| + \|y\|$ and $\|\alpha x\| = |\alpha|\|x\|$ for all $x, y \in X$ and every scalar α. If $(X, \|\cdot\|)$ is a seminormed vector space then the function $\rho : X^2 \to \mathbb{R}$,

$$\rho(x,y) = \|x - y\|,$$

is a pseudometric on X.

In the following example, we construct a strongly rigid pseudometric subspace of a seminormed real vector space.

Example 2.16. Let $(\mathbb{R}^2, \|\cdot\|)$ be a two-dimensional seminormed real vector space endowed with the seminorm $\|\cdot\|$ such that $\|\langle x,y\rangle\| = |x|$ for each $\langle x,y\rangle \in \mathbb{R}^2$. The field $\mathbb{R}$ of real numbers can be also considered as a vector space over the field $\mathbb{Q}$ of rational numbers. Let H be a linearly independent over $\mathbb{Q}$ subset of $\mathbb{R}$. If we define a subset X of $\mathbb{R}^2$ as

$$X = \{\langle x,y\rangle \in \mathbb{R}^2 : x \in H \quad \text{and} \quad y \in \mathbb{R}\},$$

then X is a strongly rigid pseudometric subspace of the seminormed space $(\mathbb{R}^2, \|\cdot\|)$.

We say that a metric $d : X^2 \to \mathbb{R}$ is *discrete* if the inequality

$$|d(X^2)| \leqslant 2$$

holds, where $|d(X^2)|$ is the cardinal number of the set $d(X^2)$.

Remark 2.17. The standard definition of *discrete metric* can be formulated as: “The metric on X is discrete if the distance from each point of X to every other point of X is one.” (See, for example, [19, p. 4].)

The following definition is a suitable reformulation of the corresponding concept from the study by Dovgoshey and Luukkainen [7].

Definition 2.18. Let (X, d) be a pseudometric space. The pseudometric d is *discrete* if all metric subspaces of (X, d) are discrete.

The next lemma is simple, and we omit the proof.

Lemma 2.19. *A pseudometric $d : X^2 \to \mathbb{R}$ is discrete if and only if $|d(X^2)| \leqslant 2$.*

Definition 2.20. A pseudometric space (X, d) is a *pseudorectangle* if all three-point metric subspaces of (X, d) are strongly rigid and isometric and, in addition, there is a four-point metric subspace Y of (X, d), such that for every $x \in X$, we can find $y \in Y$ satisfying $d(x, y) = 0$.

It is easy to see that the metric reflection $(X/\overset{0}{=}, \delta_d)$ of every pseudorectangle (X, d) is a four-point metric space and, in addition, it can be shown that this metric reflection is combinatorially similar to the vertex set of Euclidean nonsquare rectangle.

This article is organized as follows.

In the next section, we recall some known interconnections between equivalence relations, partitions of sets, and discrete pseudometrics. A simple sufficient condition for the equality $\mathbf{Cs}(X, d) = \mathbf{PI}(X, d)$ is found in Corollary 3.6 of Proposition 3.4.

The main results of the article are given in Sections 3 and 4.

A complete description of pseudometric spaces (X, d) that satisfy the equality $\mathbf{Cs}(X/\overset{0}{=}, \delta_d) = \mathrm{Sym}(X/\overset{0}{=})$ is given in Theorem 4.5. This theorem together with Corollary 3.6 allows us to characterize $\mathcal{IP}$-spaces in Theorem 4.7.

A combinatorial characterization of fibers of pseudometrics is proved in Theorems 5.3, 5.4, and 5.6 for strongly rigid spaces, discrete spaces, and pseudorectangles, respectively. As a corollary of these theorems, we obtain a new description of $\mathcal{IP}$-spaces in Theorem 5.7. In Proposition 5.9, we show that pseudorectangles or strongly rigid spaces (X, d) and (Y, ρ) are combinatorially similar if and only if the binary relations $\overset{0(d)}{=}$ and $\overset{0(\rho)}{=}$ are the same. The characteristic properties of $\overset{0}{=}$ are described in Proposition 5.10 for pseudorectangles and strongly rigid pseudometric spaces.

The final result of the article, Theorem 5.13 characterizes the classes of discrete pseudometric spaces, strongly rigid pseudometric spaces, and pseudorectangles in terms of same extremal properties of these classes.

3 Partitions of sets

Let U be a set. A *binary relation* on U is a subset of the Cartesian square

$$U^2 = U \times U = \{\langle x, y\rangle : x, y \in U\}.$$

A binary relation $R \subseteq U^2$ is an *equivalence relation* on U if the following conditions hold for all x, y, $z \in U$:

(i) $\langle x, x\rangle \in R$, the *reflexivity* property;
(ii) $(\langle x, y\rangle \in R) \Leftrightarrow (\langle y, x\rangle \in R)$, the *symmetry* property;
(iii) $((\langle x, y\rangle \in R)$ and $(\langle y, z\rangle \in R)) \Rightarrow (\langle x, z\rangle \in R)$, the *transitivity* property.

If R is an equivalence relation on U, then an *equivalence class* is a subset of U having the form

$$[a]_R = \{x \in U : \langle x, a\rangle \in R\} \tag{3.1}$$

for some $a \in U$. The *quotient set* U/R of U with respect to R is the set of all equivalence classes $[a]_R$.

Let X be a nonempty set and $P = \{X_j : j \in J\}$ be a set of nonempty subsets of X. The set P is a *partition* of X with the *blocks* X_j, $j \in J$, if $\cup_{j\in J} X_j = X$ and $X_{j_1} \cap X_{j_2} = \varnothing$ for all distinct j_1, $j_2 \in J$.

Definition 3.1. Partitions P and Q of a set X are *equal*, $P = Q$, if every block of P is a block of Q and *vice versa*.

Every partition P of a set X is a subset of the power set 2^X, $P \subseteq 2^X$, and each block of P is a point of 2^X. Thus, Definition 3.1 simply means that $P = Q$ holds if and only if P and Q are the same subsets of 2^X. Consequently, $P = Q$ holds if and only if $P \subseteq Q$ and $Q \subseteq P$. The following lemma states that any of the aforementioned inclusions is sufficient for $P = Q$.

Lemma 3.2. *Let* $P = \{X_j : j \in J\}$ *and* $Q = \{Y_i : i \in I\}$ *be partitions of a set* X. *Then the inclusion* $P \subseteq Q$ $(Q \subseteq P)$ *implies the equality*

$$P = Q. \tag{3.2}$$

Proof. Let

$$P \subseteq Q \tag{3.3}$$

hold. Then, for every $j_1 \in J$, there is $i_1 \in I$ such that $X_{j_1} = Y_{i_1}$. Suppose that inclusion (3.3) is strict. Then the set P is a proper subset of the set Q. Since every element of Q is a block Y_i, $i \in I$, there is $i_0 \in I$ such that

$$P \subseteq \{Y_i : i \in I \backslash \{i_0\}\}. \tag{3.4}$$

Now from (3.4) and the definition of partitions of sets, we obtain the contradiction,

$$X = \bigcup_{j \in J} X_j \subseteq \bigcup_{\substack{i \in I \\ i \neq i_0}} Y_i = X \backslash Y_{i_0} \subsetneq X.$$

Equality (3.2) follows. □

There exists the well known, one-to-one correspondence between the equivalence relations on sets and the partitions of sets (see, for example, [14, Chapter II, § 5] or [17, Theorem 1]).

Proposition 3.3. *Let X be a nonempty set. If* $P = \{X_j : j \in J\}$ *is a partition of X and R is a binary relation on X such that the logical equivalence*

$$(\langle x, y \rangle \in R) \Leftrightarrow (\exists j \in J : x \in X_j \text{ and } y \in X_j)$$

is valid for every $\langle x, y \rangle \in X^2$, *then R is an equivalence relation on X with the quotient set P, and it is the unique equivalence relation on X having P as the quotient set. Conversely, if R is an equivalence relation on X, then there is the unique partition P of X such that P is the quotient set of X with respect to R.*

The next proposition shows, in particular, that if $\Psi : Y \to X$ is a combinatorial similarity of pseudometric spaces (X, d) and (Y, ρ), then the equivalence classes of the relation $\overset{0(d)}{=}$ are the images of the equivalence classes of $\overset{0(\rho)}{=}$ under mapping Ψ.

Proposition 3.4. *Let* (X, d) *and* (Y, ρ) *be nonempty pseudometric spaces and let* $\Psi : Y \to X$ *be a combinatorial similarity of these spaces. If* $f : d(X^2) \to \rho(Y^2)$ *is a bijection such that*

$$\rho(x, y) = f\big(d(\Psi(x)), d(\Psi(y))\big) \tag{3.5}$$

for all $x, y \in Y$, *then the equalities*

$$f(0) = 0 \tag{3.6}$$

and

$$(X/\overset{0(d)}{=}) = \{\Psi(Y_j) : Y_j \in Y/\overset{0(\rho)}{=}\} \tag{3.7}$$

hold.

Proof. By using Definition 2.1 and equality (3.5) with $y = x$, we obtain

$$0 = \rho(x, x) = f\big(d(\Psi(x)), d(\Psi(x))\big) = f(0)$$

that implies (3.6).

To prove equality (3.7), we note that $\{\Psi(Y_j) : Y_j \in Y/\overset{0(\rho)}{=}\}$ is a partition of the set X, because $\Psi : Y \to X$ is bijective and $Y/\overset{0(\rho)}{=}$ is a partition of Y by Proposition 2.9. Now Proposition 3.3 and (2.3) imply that (3.7) holds if and only if

$$\big(d(x,y) = 0\big) \Leftrightarrow \big(\rho(\Psi^{-1}(x), \Psi^{-1}(y)) = 0\big) \tag{3.8}$$

for all x, $y \in X$. Logical equivalence (3.8) is valid for all x, $y \in X$ if and only if

$$\big(d(\Psi(u), \Psi(v)) = 0\big) \Leftrightarrow \big(\rho(u, v) = 0\big) \tag{3.9}$$

for all $u, v \in Y$. Since $f : d(X^2) \to \rho(Y^2)$ is bijective, equality (3.6) implies that $d(\Psi(u), \Psi(v)) = 0$ holds if and only if $f(d(\Psi(u), \Psi(v))) = 0$. Hence, (3.8) can be written as follows:

$$\big(f(d(\Psi(u), \Psi(v))) = 0\big) \Leftrightarrow \big(\rho(u, v) = 0\big). \tag{3.10}$$

Now the validity of (3.10) follows from (3.5). ☐

For the case $(X, d) = (Y, \rho)$, Proposition 3.4 implies that the combinatorial self-similarities preserve the equivalence relation $\overset{0(d)}{=}$.

Corollary 3.5. *Let d and ρ be two combinatorially similar pseudometrics defined on the same nonempty set. Then the binary relations $\overset{0(d)}{=}$ and $\overset{0(\rho)}{=}$ are equal.*

The next corollary gives a simple sufficient condition under which equality (2.5) holds.

Corollary 3.6. *Let (X, d) be a nonempty pseudometric space, and let $\{X_j : j \in J\}$ be a partition of X corresponding the equivalence relation $\overset{0(d)}{=}$. If distinct blocks of this partition have different numbers of points, $|X_{j_1}| \neq |X_{j_2}|$ for different $j_1, j_2 \in J$, then the equality*

$$\mathbf{Cs}(X, d) = \mathbf{PI}(X, d)$$

holds.

Proof. Let

$$|X_{j_1}| \neq |X_{j_2}| \tag{3.11}$$

holds for all different $j_1, j_2 \in J$. Let us consider an arbitrary combinatorial self-similarity $\Psi : X \to X$ of (X, d). We must show that Ψ is a pseudoidentity of (X, d). To do it, we rewrite equality (3.7) in the form

$$\{X_j : j \in J\} = \{\Psi(X_j) : j \in J\}. \tag{3.12}$$

Since Ψ is a bijective mapping, $|\Psi(X_j)| = |X_j|$ holds for every $j \in J$. Now, $\Psi \in \mathbf{PI}(X, d)$ follows from (3.11) and (3.12). ☐

The next result directly follows from Proposition 3.6 and Corollary 3.7 of the study by Dovgoshey and Luukkainen [7], and shows that a partial converse to Proposition 2.9 is also true.

Proposition 3.7. *Let X be a nonempty set and let $\equiv$ be an equivalence relation on X. Then there is a unique up to combinatorial similarity discrete pseudometric $d : X^2 \to \mathbb{R}$ such that*

$$(x \equiv y) \Leftrightarrow (d(x,y) = 0) \tag{3.13}$$

is valid for all x, $y \in X$.

Corollary 3.8. *Let d and ρ be discrete pseudometrics on a set X. Then the following statements are equivalent:*

(*i*) *The pseudometric spaces (X, d) and (X, ρ) are combinatorially similar.*

(*ii*) *The binary relations $\overset{0(d)}{=}$ and $\overset{0(\rho)}{=}$ are the same.*

In the following section of this article, we will prove that the equality of binary relations $\overset{0(d)}{=}$ and $\overset{0(\rho)}{=}$ is equivalent to combinatorial similarity of (X, d) and (X, ρ), when both spaces are pseudorectangles or strongly rigid pseudometric spaces.

4 Structure of $\mathcal{IP}$-spaces

The following theorem is a special case of Theorem 2.8 [2], which gives us a complete description of semimetric spaces satisfying the equality

$$\mathbf{Cs}(X, d) = \mathrm{Sym}(X).$$

Theorem 4.1. *Let (X, d) be a nonempty metric space. Then the following statements are equivalent:*

(*i*) *At least one of the following conditions has been fulfilled:*

  (i_1) *(X, d) is strongly rigid;*

  (i_2) *(X, d) is discrete;*

  (i_3) *All three-point subspaces of (X, d) are strongly rigid and isometric.*

(*ii*) *The equality $\mathbf{Cs}(X, d) = \mathrm{Sym}(X)$ holds.*

Our initial objective is to find a “pseudometric” analog of this theorem.

The next lemma shows that the transition from pseudometric space to its metric reflection preserves the concept of discreteness, strong rigidity, and the property of being a pseudorectangle.

Lemma 4.2. *Let (X, d) be a nonempty pseudometric space. Then the following statements hold:*

(*i*) *(X, d) is strongly rigid if and only if $(X/\overset{0}{=}, \delta_d)$ is strongly rigid.*

(*ii*) *(X, d) is discrete if and only if $(X/\overset{0}{=}, \delta_d)$ is discrete.*

(*iii*) *(X, d) is a pseudorectangle if and only if $(X/\overset{0}{=}, \delta_d)$ contains exactly four points, and all three-point subspaces of $(X/\overset{0}{=}, \delta_d)$ are strongly rigid and isometric.*

Proof. Let (X, d) be strongly rigid. Then, by using the Axiom of Choice (AC), we find a metric subspace Y of the pseudometric space (X, d) such that for every $x \in X$ there is the unique $y \in Y$ that satisfies $d(x, y) = 0$. Since (X, d) is strongly rigid, the metric space Y is also strongly rigid by Definition 2.13. Moreover, Proposition 2.9 implies that the metric spaces Y and $(X/\overset{0}{=}, \delta_d)$ are isometric. Hence, $(X/\overset{0}{=}, \delta_d)$ is strongly rigid.

To complete the proof of statement (*i*) we must show that the strong rigidity of $(X/\overset{0}{=}, \delta_d)$ implies that (X, d) is also a strongly rigid.

Let $(X/\overset{0}{=}, \delta_d)$ be strongly rigid, and let Z be a metric subspace of (X, d). By using AC, we find a metric subspace Y_Z of (X, d) such that $Y_Z \supseteq Z$, and for every $x \in X$, there is the unique $y \in Y_Z$, which satisfies $d(x, y) = 0$. Proceeding as earlier, we can see that Y_Z and $(X/\overset{0}{=}, \delta_d)$ are isometric. Hence, Y_Z is strongly rigid and, consequently, being a subspace of Y_X, Z is also strongly rigid.

Statement (*i*) follows.

Let us prove statement (*ii*). Let $\pi : X \to X/\overset{0}{=}$ be the canonical projection,

$$\pi(x) \coloneqq \{y \in X : d(x, y) = 0\}.$$

Then, by formula (2.4) of Proposition 2.9, the equality

$$d(x,y) = \delta_d(\pi(x), \pi(y)) \tag{4.1}$$

holds for all $x, y \in X$. By using (4.1), we see that d and δ_d has one and the same range, and, consequently (*ii*) holds by Lemma 2.19.

The validity of statement (*iii*) follows from Proposition 2.9 and Definition 2.20. ☐

The following lemma is, in fact, a particular case of Proposition 2.3 from the study by Bilet and Dovgoshey [2].

Lemma 4.3. *Let* (X, d) *be a metric space with* $|X| \geqslant 4$. *Then the following statements are equivalent*:

(i) *All three-point subspaces of* (X, d) *are strongly rigid and isometric.*

(ii) (X, d) *is combinatorially similar to the space of vertices of Euclidean nonsquare rectangle.*

Example 4.4. The four-point subspace $Y = \{p, q, l, m\}$ of the complex plane $\mathbb{C}$ with $p = 0, q = 3i, l = 4 + 3i$, and $m = 4$ is a Euclidean nonsquare rectangle (Figure 1).

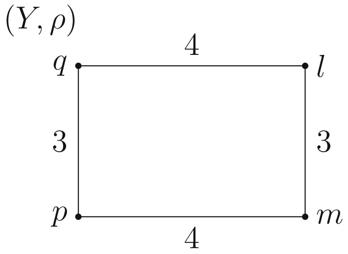

Figure 1: If ρ is the restriction of the usual Euclidean metric on Y, then every metric space (X, d) satisfying $|X| \geqslant 4$ and condition (*i*) of Lemma 4.3 is combinatorially similar to (Y, ρ).

Now we are ready to characterize the pseudometric spaces satisfying equality (2.6).

Theorem 4.5. *Let* (X, d) *be a nonempty pseudometric space. Then the following statements are equivalent*:

(*i*) *At least one of the following conditions has been fulfilled*:
  - (i_1) (X, d) *is strongly rigid*;
  - (i_2) (X, d) *is discrete*;
  - (i_3) (X, d) *is a pseudorectangle.*

(*ii*) *The equality*

$$\mathbf{Cs}(X/\overset{0}{=}, \delta_d) = \mathrm{Sym}(X/\overset{0}{=}) \tag{4.2}$$

holds.

Proof. (i) ⇒ (ii). Let statement (*i*) hold. Then, by Lemma 4.2, at least one from the following statements is valid:

(s_1) $(X/\overset{0}{=}, \delta_d)$ is strongly rigid;

(s_2) $(X/\overset{0}{=}, \delta_d)$ is discrete;

(s_3) All three-point subspaces of $(X/\overset{0}{=}, \delta_d)$ are strongly rigid and isometric and, in addition, $|X/\overset{0}{=}| = 4$ holds.

Now (4.2) follows from Theorem 4.1.

(ii) ⇒ (i). Let equality (4.2) hold. Then, by Theorem 4.1, we have (s_1) or (s_2), or

(s_4) All three-point subspaces of $(X/\overset{0}{=}, \delta_d)$ are strongly rigid and discrete.

By Lemma 4.2, statements (s_1) and (s_2) imply, respectively, statements (i_1) and (i_2) of the theorem being proved. By using Lemma 4.3, we see that statements (s_4) and (s_3) are equivalent. Consequently, (s_4) implies (i_3) by Lemma 4.2. □

The next proposition describes the combinatorial self-similarities of a pseudometric space (X, d) via combinatorial self-similarities of the metric reflection $(X/\overset{0}{=}, \delta_d)$ of this space.

Proposition 4.6. *Let (X, d) be a pseudometric space, let $\Phi : X \to X$ be a bijective mapping and let $\pi : X \to X/\overset{0}{=}$ be the canonical projection,*

$$\pi(x) = \{y \in X : d(x, y) = 0\}.$$

If there is a combinatorial self-similarity $\Psi : X/\overset{0}{=} \to X/\overset{0}{=}$ of $(X/\overset{0}{=}, \delta_d)$ such that diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\Phi"'] & X/\overset{0}{=} \arrow[d, "\Psi"] \\ X \arrow[r, "\pi"] & X/\overset{0}{=} \end{tikzcd} \tag{4.3}$$

is commutative, then Φ is a combinatorial self-similarity of (X, d).

Proof. Let Ψ be a combinatorial self-similarity of $(X/\overset{0}{=}, \delta_d)$. Then, by using Definition 2.2, we find a bijection $f : \delta_d(X/\overset{0}{=})^2 \to \delta_d(X/\overset{0}{=})^2$ such that the following diagram

$$\begin{tikzcd} (X/\overset{0}{=})^2 \arrow[r, "\delta_d"] \arrow[d, "\Psi \otimes \Psi"'] & \delta_d(X/\overset{0}{=})^2 \arrow[d, "f"] \\ (X/\overset{0}{=})^2 \arrow[r, "\delta_d"] & \delta_d(X/\overset{0}{=})^2 \end{tikzcd} \tag{4.4}$$

is commutative, where

$$\Psi \otimes \Psi(\langle a, b\rangle) = \langle \Psi(a), \Psi(b)\rangle$$

for every $\langle a, b\rangle \in (X/\overset{0}{=})^2$. Suppose diagram (4.3) is commutative. Then

$$\begin{tikzcd} X^2 \arrow[r, "\pi\otimes\pi"] \arrow[d, "\Phi \otimes \Phi"'] & (X/\overset{0}{=})^2 \arrow[d, "\Psi \otimes \Psi"] \\ X^2 \arrow[r, "\pi\otimes\pi"] & (X/\overset{0}{=})^2 \end{tikzcd} \tag{4.5}$$

also is a commutative diagram, where

$$\Phi \otimes \Phi(\langle x, y\rangle) = \langle \Phi(x), \Phi(y)\rangle$$

for every $\langle x, y\rangle \in X^2$. The commutativity of (4.4) and (4.5) implies that

$$\begin{tikzcd} X^2 \arrow[r, "\pi\otimes\pi"] \arrow[d, "\Phi \otimes \Phi"'] & (X/\overset{0}{=})^2 \arrow[r, "\delta_d"] \arrow[d, "\Psi \otimes \Psi"] & \delta_d(X/\overset{0}{=})^2 \arrow[d, "f"] \\ X^2 \arrow[r, "\pi\otimes\pi"] & (X/\overset{0}{=})^2 \arrow[r, "\delta_d"] & \delta_d(X/\overset{0}{=})^2 \end{tikzcd} \tag{4.6}$$

is commutative. By Proposition 2.9, we have

$$\delta_d(X/\overset{0}{=})^2 = d(X^2)$$

and, in addition, this proposition implies the equality of mappings $X^2 \overset{d}{\to} d(X^2)$ and

$$X^2 \overset{\pi\otimes\pi}{\to} (X/\overset{0}{=})^2 \overset{\delta_d}{\to} \delta_d(X/\overset{0}{=})^2.$$

Hence, the commutativity of (4.6) gives us the commutativity of the diagram

$$\begin{tikzcd} X^2 \arrow[r, "d"] \arrow[d, "\Phi\otimes\Phi"'] & d(X^2) \arrow[d, "f"] \\ X^2 \arrow[r, "d"] & d(X^2). \end{tikzcd}$$

By Definition 2.2, the last diagram is commutative iff $\Phi : X \to X$ is a combinatorial self-similarity of (X, d). ☐

The next theorem is one of the main results of the article.

Theorem 4.7. *Let (X, d) be a nonempty pseudometric space, and let $\{X_j : j \in J\}$ be a partition of X corresponding the equivalence relation $\overset{0(d)}{=}$. Then $(X, d) \in \mathcal{IP}$ if and only if*

$$|X_{j_1}| \neq |X_{j_2}| \tag{4.7}$$

holds whenever $j_1, j_2 \in J$ are distinct and, in addition, at least one of the following conditions has been fulfilled:
- (*i*) *(X, d) is strongly rigid;*
- (*ii*) *(X, d) is discrete;*
- (*iii*) *(X, d) is a pseudorectangle.*

Proof. Suppose that (4.7) holds whenever $j_1, j_2 \in J$ are distinct, and that al least one from conditions (*i*) − (*iii*) has been fulfilled. Then the membership $(X, d) \in \mathcal{IP}$ follows from Corollary 3.6 and Theorem 4.5.

Let (X, d) belong to $\mathcal{IP}$. Then the equality

$$\mathbf{Cs}(X/\overset{0}{=}, \delta_d) = \mathrm{Sym}(X/\overset{0}{=}) \tag{4.8}$$

holds and, consequently, at least one from conditions (*i*)–(*iii*) is valid. Thus, to complete the proof, it suffices to show that (4.7) is valid for all distinct $j_1, j_2 \in J$.

Suppose contrary that there exist $j_1, j_2 \in J$ such that $j_1 \neq j_2$ but $|X_{j_1}| = |X_{j_2}|$. Then there is a bijection $\Phi : X \to X$, which satisfies the equalities

$$\Phi(X_{j_1}) = X_{j_2} \quad \text{and} \quad \Phi(X_j) = X_j \tag{4.9}$$

whenever $j \in J$ and $j_1 \neq j \neq j_2$.

Let x_{j_1} and x_{j_2} be some points of X_{j_1} and X_{j_2}, respectively. Write

$$x_1^* = \pi(x_{j_1}) \quad \text{and} \quad x_2^* = \pi(x_{j_2}), \tag{4.10}$$

where π is the canonical projection of X on $X/\overset{0}{=}$ and define a bijection $\Psi : X/\overset{0}{=} \to X/\overset{0}{=}$ as follows:

$$\Psi(x) := \begin{cases} x_1^* & \text{if } x = x_2^*, \\ x_2^* & \text{if } x = x_1^*, \\ x & \text{otherwise}. \end{cases} \tag{4.11}$$

It follows from (4.9) to (4.11) that the diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\Phi"'] & X/\overset{0}{=} \arrow[d, "\Psi"] \\ X \arrow[r, "\pi"] & X/\overset{0}{=} \end{tikzcd}$$

is commutative. Moreover, the mapping Ψ is a combinatorial self-similarity of $(X/\overset{0}{=}, \delta_d)$ by (4.8). Hence, Φ is a combinatorial self-similarity of (X, d) by Proposition 4.6. The equality $d(x_1^*, \Phi(x_1^*)) = d(x_1^*, x_2^*)$

and the inequality $d(x_1^*, x_2^*) > 0$ imply $\Phi \notin \mathbf{PI}(X, d)$. Thus, we have $\Phi \in \mathbf{Cs}(X, d) \backslash \mathbf{PI}(X, d)$, contrary to $(X, d) \in \mathcal{IP}$. □

We conclude this section with the following open problem closely related to Theorems 4.5 and 4.7.

Problem 4.8. Describe the structure of pseudometric spaces (X, d) for which

$$\mathbf{Cs}(X, d) = \mathbf{PI}(X, d).$$

5 From partitions of X to partitions of X^2

It is well known that for every nonempty set X and arbitrary surjection $f : X \to Y$, the family

$$P_{f^{-1}} := \{f^{-1}(y) : y \in Y\}$$

is a partition of X, where $f^{-1}(y)$ is the inverse image of the singleton $\{y\}$,

$$f^{-1}(y) := \{x \in X : f(x) = y\}.$$

In what follows, we set

$$P_{d^{-1}} := \{d^{-1}(t) : t \in d(X^2)\} \tag{5.1}$$

for every nonempty pseudometric space (X, d).

In this section, we describe the structure of the partition $P_{d^{-1}}$, when (X, d) is strongly rigid or discrete, or (X, d) is a pseudorectangle. This allows us to obtain new characteristics of $\mathcal{IP}$-spaces and expand Corollary 3.8 to strongly rigid pseudometric spaces and pseudorectangles.

The following lemma gives a “constructive variant” of Proposition 3.3.

Lemma 5.1. *Let X be a nonempty set, and let $P = \{X_j : j \in J\}$ be a partition of X. If R is the equivalence relation corresponding to P, then the equality $R = \cup_{j\in J} X_j^2$ holds.*

For the proof of Lemma 5.1 see, for example, Theorem 6 in [13, p. 9].

Proposition 5.2. *Let (X, d) be a pseudometric space, and $\{X_j : j \in J\}$ be the quotient set of X with respect to the equivalence relation $\overset{0}{=}$. Then for any fixed nonzero element t_0 of $d(X^2)$, the following statements are equivalent:*

(*i*) *There are different $j_1, j_2 \in J$ such that*

$$d^{-1}(t_0) = (X_{j_1} \times X_{j_2}) \cup (X_{j_2} \times X_{j_1}). \tag{5.2}$$

(*ii*) *The assertion*

$$\big((x \overset{0}{=} u) \text{ and } (y \overset{0}{=} v)\big) \text{ or } \big((x \overset{0}{=} v) \text{ and } (y \overset{0}{=} u)\big) \tag{5.3}$$

is valid whenever

$$d(x, y) = t_0 = d(u, v). \tag{5.4}$$

Proof. (i) ⇒ (ii). Let different $j_1, j_2 \in J$ satisfy (5.2). We must show that (5.4) implies (5.3) for all $x, y, u, v \in X$.

Suppose (5.4) holds. Then we have

$$\langle x, y\rangle, \langle u, v\rangle \in d^{-1}(t_0).$$

Since the sets $X_{j_1} \times X_{j_2}$ and $X_{j_2} \times X_{j_1}$ are disjoint, (5.2) implies that only the following cases are possible:

$$\langle x, y\rangle \in X_{j_1} \times X_{j_2} \text{ and } \langle u, v\rangle \in X_{j_1} \times X_{j_2}, \tag{5.5}$$

$$\langle x, y\rangle \in X_{j_2} \times X_{j_1} \text{ and } \langle u, v\rangle \in X_{j_2} \times X_{j_1}, \tag{5.6}$$

$$\langle x, y\rangle \in X_{j_1} \times X_{j_2} \text{ and } \langle u, v\rangle \in X_{j_2} \times X_{j_1}, \tag{5.7}$$

$$\langle x, y\rangle \in X_{j_2} \times X_{j_1} \text{ and } \langle u, v\rangle \in X_{j_1} \times X_{j_2}. \tag{5.8}$$

The sets X_{j_1} and X_{j_2} are different elements of the quotient set $X/\overset{0}{=}$. Hence, each of (5.5) and (5.6) implies $x \overset{0}{=} u$ and $y \overset{0}{=} v$. Analogously, $x \overset{0}{=} v$ and $y \overset{0}{=} u$ hold whenever (5.7) or (5.8) is valid. Thus, (ii) holds.

(ii) ⇒ (i). Let (ii) hold and let x, y be points of X satisfying

$$d(x, y) = t_0. \tag{5.9}$$

Since $\{X_j : j \in J\}$ is the quotient set of X with respect to $\overset{0}{=}$, there are $j_1^0, j_2^0 \in J$ such that $x \in X_{j_1^0}$ and $y \in X_{j_2^0}$. Equality (5.9) and the inequality $t_0 > 0$ imply that $j_1^0 \neq j_2^0$.

We claim that (5.2) holds with $j_1 = j_1^0$ and $j_2 = j_2^0$. Indeed, the inclusion

$$d^{-1}(t_0) \supseteq (X_{j_1^0} \times X_{j_2^0}) \cup (X_{j_2^0} \times X_{j_1^0})$$

follows from the triangle inequality, the symmetric property of d, and the definition of $\overset{0}{=}$ (see (2.3)). Hence, to prove (5.2), we must show that the membership

$$\langle u, v\rangle \in (X_{j_1^0} \times X_{j_2^0}) \cup (X_{j_2^0} \times X_{j_1^0}) \tag{5.10}$$

is valid whenever

$$d(u, v) = t_0. \tag{5.11}$$

To complete the proof, it suffices to note that (5.9) and (5.11) imply (5.3), (5.4), and that (5.3) implies (5.10). □

Comparing Definition 2.13 with statement (ii) of Proposition 5.2, we see that (ii) can be considered as a singular version of the global property "to be strongly rigid." In Theorem 5.3, we characterize the strong rigidity of pseudometrics by "globalization" of statement (i) of Proposition 5.2.

Let $Q = \{X_j : j \in J\}$ be a partition of a nonempty set X. Then we define a partition $Q \otimes_1 Q$ of X^2 by the rule: "If $|J| = 1$, then $Q \otimes_1 Q := \{X^2\}$, otherwise $B \subseteq X^2$ is a block of $Q \otimes_1 Q$ if and only if either $B = \bigcup_{j\in J} X_j^2$ or there are *distinct* $j_1, j_2 \in J$ such that $B = \big(X_{j_1} \times X_{j_2}\big) \cup \big(X_{j_2} \times X_{j_1}\big)$."

The next theorem follows directly from Theorem 4.13 and Corollary 4.14 of [7].

Theorem 5.3. *Let (X, d) be a nonempty pseudometric space. Then the following conditions are equivalent*:

(*i*) *$d : X^2 \to \mathbb{R}$ is strongly rigid.*

(*ii*) *If Q is a partition corresponding to $\overset{0}{=}$, then $Q \otimes_1 Q$ and $P_{d^{-1}}$ are equal,*

$$Q \otimes_1 Q = P_{d^{-1}}. \tag{5.12}$$

(*iii*) *There is a partition Q of X such that* (5.12) *holds.*

Similarly to $Q \otimes_1 Q$ for every partition of $Q = \{X_j : j \in J\}$ of X we define $Q \otimes_2 Q$ as: "If $|J| = 1$, then $Q \otimes_2 Q := \{X^2\}$, otherwise a set $B \subseteq X^2$ is a block of $Q \otimes_2 Q$ if and only if either $B = \bigcup_{j\in J} X_j^2$ or $B = X^2 \backslash \bigcup_{j\in J} X_j^2$."

The following result is an analog of Theorem 5.3 for discrete pseudometrics.

Theorem 5.4. *Let (X, d) be a nonempty pseudometric space. Then the following conditions are equivalent*:

(*i*) *$d : X^2 \to \mathbb{R}$ is discrete.*

(*ii*) *If Q is a partition corresponding to $\overset{0}{=}$, then $Q \otimes_2 Q$ and $P_{d^{-1}}$ are equal,*

$$Q \otimes_2 Q = P_{d^{-1}}. \tag{5.13}$$

(*iii*) *There is a partition Q of X such that* (5.13) *holds.*

Proof. The implications (*i*) ⇒ (*ii*) ⇒ (*iii*) ⇒ (*i*) are evidently valid if d is the zero pseudometric on X. Let us consider the case when $|d(X^2)| \geqslant 2$.

(*i*) ⇒ (*ii*). Let d be a discrete pseudometric, and let

$$Q = \{X_j : j \in J\}$$

be the partition of X corresponding to the relation $\overset{0(d)}{=}$. By Lemma 5.1, the equality

$$(\overset{0(d)}{=}) = \bigcup_{j \in J} X_j^2$$

holds. By using (2.3), we see that

$$d^{-1}(0) = \bigcup_{j \in J} X_j^2. \tag{5.14}$$

By Lemma 2.19, the inequality $|d(X^2)| \leqslant 2$ holds for discrete d. The last inequality and $|d(X^2)| \geqslant 2$ imply that $|d(X^2)| = 2$. Thus, the partition $P_{d^{-1}}$ of X^2 contains exactly two blocks. Since one of this block is $d^{-1}(0)$, the second one coincides with $X^2 \setminus \bigcup_{j \in J} X_j^2$ by (5.14). Now (*ii*) follows from the definition of $Q \otimes_2 Q$.

(*ii*) ⇒ (*iii*). This implication is trivially valid.

(*iii*) ⇒ (*i*). Let a partition $Q = \{X_j : j \in J\}$ of X satisfy equality (5.13). Then this equality and the definition of $Q \otimes_2 Q$ imply $|d(X^2)| = 2$. By using Lemma 2.19, we see that d is a discrete pseudometric. □

Remark 5.5. Theorem 5.4 can be considered as a special case of Theorem 3.9 from the study by Dovgoshey and Luukkainen [7], which describes all mappings with domain X^2, which are combinatorially similar to discrete pseudometrics on X.

Let X be a set with $|X| \geqslant 4$ and let $Q = \{X_1, X_2, X_3, X_4\}$ be a partition of the set X. Let us denote by $Q \otimes_3 Q$ a partition of X^2 having the four blocks: $\bigcup_{i=1}^4 X_i^2$,

$$(X_1 \times X_2) \cup (X_2 \times X_1) \cup (X_3 \times X_4) \cup (X_4 \times X_3), \tag{5.15}$$

$$(X_1 \times X_3) \cup (X_3 \times X_1) \cup (X_2 \times X_4) \cup (X_4 \times X_2), \tag{5.16}$$

$$(X_1 \times X_4) \cup (X_4 \times X_1) \cup (X_2 \times X_3) \cup (X_3 \times X_2). \tag{5.17}$$

Theorem 5.6. *Let (X, d) be a nonempty pseudometric space. Then the following conditions are equivalent*:

(*i*) *(X, d) is a pseudorectangle.*

(*ii*) *If Q is a partition corresponding to $\overset{0}{=}$, then $Q \otimes_3 Q$ and $P_{d^{-1}}$ are equal,*

$$Q \otimes_3 Q = P_{d^{-1}}. \tag{5.18}$$

(*iii*) *There is a partition Q of X such that* (5.18) *holds.*

Proof. (*i*) ⇒ (*ii*). Let (X, d) be a pseudorectangle.

By Definition 2.20, there is a set $Y = \{y_1, y_2, y_3, y_4\}$, such that $Y \subseteq X$ and the sets $[y_i]_{\overset{0}{=}}$,

$$[y_i]_{\overset{0}{=}} = \{x \in X : d(x, y_i) = 0\}, \ i = 1, \ldots, 4,$$

are the equivalence classes of the relation $\overset{0}{=}$.

We claim that (5.18) holds if

$$Q = \{X_1, X_2, X_3, X_4\} \tag{5.19}$$

with

$$X_i = [y_i]_{\overset{0}{=}}, \quad i = 1, \dots, 4. \tag{5.20}$$

To prove the last claim, we first show that $P_{d^{-1}}$ contains exactly four blocks, i.e.,

$$|d(X^2)| = 4 \tag{5.21}$$

holds.

Indeed, by Definition 2.20, all three-point metric subspaces of (X, d) are isometric that implies

$$|d(X^2)| \leqslant 4. \tag{5.22}$$

Moreover, it is easy to see that a finite nonempty metric space (Z, ρ) is strongly rigid if and only if the number of two-point subsets of Z is the same as the number of nonzero elements of $\rho(Z^2)$,

$$|\rho(Z^2)| = \frac{|Z||Z-1|}{2} + 1.$$

In particular, a three-point metric subspace S of the pseudometric space (X, d) is strongly rigid if and only if $|d(S^2)| = 4$. Consequently, Definition 2.20 implies

$$|d(X^2)| \geqslant |d(S^2)| = 4 \tag{5.23}$$

whenever S is a three-point metric subspace of (X, d). Now, equality (5.21) follows from (5.22) and (5.23).

Let us prove (5.18). By Lemma 3.2, it suffices to show that the inclusion

$$P_{d^{-1}} \subseteq Q \ \otimes_3 \ Q$$

holds, i.e., that

$$d^{-1}(t) \in Q \ \otimes_3 \ Q \tag{5.24}$$

is valid for every $t \in d(X^2)$.

Let $t_1 \in d(X^2)$ and $t_1 > 0$ hold. Then there exist two different points $y_{i_1}, y_{i_2} \in \{y_1, y_2, y_3, y_4\}$ such that $d(y_{i_1}, y_{i_2}) = t_1$. Without loss the generality, we assume that $i_1 = 1$ and $i_2 = 2$. Then, by (5.20), we have

$$y_{i_1} = y_1 \in X_1 \quad \text{and} \quad y_{i_2} = y_2 \in X_2,$$

that implies

$$d^{-1}(t_1) \supseteq (X_1 \times X_2) \cup (X_2 \times X_1).$$

All triangles of the metric space $\{y_1, y_2, y_3, y_4\}$ are isometric and strongly rigid according to Lemma 4.3. By using this fact and comparing the four triangles $\{y_1, y_2, y_3\}$, $\{y_1, y_2, y_4\}$, $\{y_1, y_3, y_4\}$, $\{y_2, y_3, y_4\}$ we see that the equality $d(y_1, y_2) = t_1$ implies the equality

$$d(y_3, y_4) = t_1, \tag{5.25}$$

and, in addition,

$$d(y_1, y_3) = d(y_2, y_4) \neq t_1 \neq d(y_1, y_4) = d(y_2, y_3). \tag{5.26}$$

Now equality (5.25) implies

$$d^{-1}(t_1) \supseteq (X_3 \times X_4) \cup (X_4 \times X_3),$$

and, consequently,

$$d^{-1}(t_1) \supseteq (X_1 \times X_2) \cup (X_2 \times X_1) \cup (X_3 \times X_4) \cup (X_4 \times X_3).$$

If the last inclusion is strict, then there are points $x_1, x_2 \in X$ such that $d(x_1, x_2) = t_1$ and

$$\langle x_1, x_2 \rangle \in (X_1 \times X_3) \cup (X_3 \times X_1) \cup (X_2 \times X_4) \cup (X_4 \times X_2) \cup$$
$$(X_1 \times X_4) \cup (X_4 \times X_1) \cup (X_2 \times X_3) \cup (X_3 \times X_2)$$

are satisfied, contrary to (5.26). Thus, the equality

$$d^{-1}(t_1) = (X_1 \times X_2) \cup (X_2 \times X_1) \cup (X_3 \times X_4) \cup (X_4 \times X_3)$$

holds, that together with

$$d^{-1}(0) = \bigcup_{i=1}^{4} X_i^2$$

implies (5.24) for every $t \in d(X^2)$.

(*ii*) ⇒ (*iii*). This implication is trivially valid.

(*iii*) ⇒ (*i*). Let equality (5.18) hold with $Q = \{X_1, X_2, X_3, X_4\}$. We must prove that (X, d) is a pseudorectangle.

By Lemma 5.1, the equality

$$d^{-1}(0) = \bigcup_{i=1}^{4} X_i^2 \tag{5.27}$$

holds. Moreover, the definitions of $Q \otimes_3 Q$ and (5.18) imply the equality

$$|d(X^2)| = 4. \tag{5.28}$$

Let us consider a four-point set $Y = \{y_1, y_2, y_3, y_4\}$ such that $y_i \in X_i$ holds for every $i \in \{1, 2, 3, 4\}$. Then, by using (5.27), we see that Y is a four-point metric subspace of (X, d) and, for every $x \in X$, there is $y \in Y$ such that $d(x, y) = 0$. Now, by Definition 2.20, (X, d) is a pseudorectangle if and only if all three-point metric subspaces of (X, d) are strongly rigid and isometric.

Hence, by using (5.28), we see that (X, d) is a pseudorectangle if and only if

$$|d(Z^2)| = 4 \tag{5.29}$$

holds for every three-point metric subspace Z of (X, d). Let us consider arbitrary $z_1, z_2, z_3 \in X$ such that $d(z_i, z_j) \neq 0$ for all distinct $i, j \in \{1, 2, 3\}$. Since every permutation

$$\{X_1, X_2, X_3, X_4\} \to \{X_1, X_2, X_3, X_4\}$$

preserves the partition $Q \otimes_3 Q$ of the set X^2, i.e., $Q \otimes_3 Q = \widetilde{Q} \otimes_3 \widetilde{Q}$, where $\widetilde{Q}$ is a permuted partition of X, we can assume that $z_1 \in X_1, z_2 \in X_2$ and $z_3 \in X_3$. Now (5.29) follows from (5.15), (5.16), and (5.17). □

By using Theorem 5.3 and Theorems 5.4 and 5.6, we obtain the following modification of Theorem 4.7.

Theorem 5.7. *Let (X, d) be a nonempty pseudometric space, and let $Q = \{X_j : j \in J\}$ be a partition of X corresponding to the equivalence relation $\overset{0(d)}{=}$. Then $(X, d) \in \mathcal{IP}$ if and only if*

$$P_{d^{-1}} \in \{Q \otimes_1 Q, Q \otimes_2 Q, Q \otimes_3 Q\}$$

and $|X_{j_1}| \neq |X_{j_2}|$ holds whenever $j_1, j_2 \in J$ are distinct.

Let us extend Corollary 3.8 to pseudorectangles and strongly rigid pseudometric spaces.

Lemma 5.8. *Let (X, d) and (X, ρ) be nonempty pseudometric spaces. If the equality*

$$P_{d^{-1}} = P_{\rho^{-1}} \tag{5.30}$$

holds, then the identical mapping $Id_X : X \to X$ is a combinatorial similarity of (X, d) and (X, ρ).

Proof. Let (5.30) hold. Then by (5.1), we have

$$\{d^{-1}(t) : t \in d(X^2)\} = \{\rho^{-1}(\tau) : \tau \in \rho(X^2)\}$$

and, consequently, there is a bijection $f\colon d(X^2) \to \rho(X^2)$ such that

$$f(t) = \tau \quad \text{if and only if} \quad \rho^{-1}(\tau) = d^{-1}(t) \tag{5.31}$$

whenever $t \in d(X^2)$, $\tau \in \rho(X^2)$.

By using (5.31), it is easy to see that the diagram

$$\begin{tikzcd}
X^2 \arrow[r, "d"] \arrow[d, "Id_{X^2}"'] & d(X^2) \arrow[d, "f"] \\
X^2 \arrow[r, "\rho"] & \rho(X^2).
\end{tikzcd} \tag{5.32}$$

is commutative, when $Id_{X^2} : X^2 \to X^2$ is the identical mapping of X^2.

The mapping Id_{X^2} coincides with the mapping $Id_X \otimes Id_X$,

$$Id_{X^2}(\langle x, y\rangle) = \langle x, y\rangle = \langle Id_X(x), Id_X(y)\rangle$$

holds for every $\langle x, y\rangle \in X^2$. Thus, the commutativity of (5.32) implies the commutativity of

$$\begin{tikzcd}
X^2 \arrow[r, "d"] \arrow[d, "Id_X \otimes Id_X"'] & d(X^2) \arrow[d, "f"] \\
X^2 \arrow[r, "\rho"] & \rho(X^2).
\end{tikzcd}$$

By Definition 2.2, the last diagram is commutative if and only if the mapping $Id_X : X \to X$ is a combinatorial similarity of (X, d) and (X, ρ). ☐

Proposition 5.9. *Let (X, d) and (X, ρ) be either pseudorectangles or nonempty strongly rigid pseudometric spaces. Then the following statements are equivalent*:

(*i*) *The pseudometric spaces (X, d) and (X, ρ) are combinatorially similar.*

(*ii*) *The binary relations $\overset{0(d)}{=}$ and $\overset{0(\rho)}{=}$ are the same.*

Proof. $(i) \Rightarrow (ii)$. The validity of this implication follows from Corollary 3.5.

$(ii) \Rightarrow (i)$. Let (ii) hold. Suppose that both spaces (X, d) and (X, ρ) are strongly rigid.

Let $Q = \{X_j : j \in J\}$ be the partition of X corresponding the equivalence relation $\overset{0(d)}{=}$. Then, by Theorem 5.3, we have

$$P_{d^{-1}} = Q \ \otimes_1 \ Q. \tag{5.33}$$

Since the relations $\overset{0(d)}{=}$ and $\overset{0(\rho)}{=}$ are the same, we have

$$P_{\rho^{-1}} = Q \ \otimes_1 \ Q. \tag{5.34}$$

Equalities (5.33) and (5.34) imply the equality

$$P_{d^{-1}} = P_{\rho^{-1}}.$$

Consequently, (X, d) and (X, ρ) are combinatorially similar by Lemma 5.8.

For the case when (X, d) and (X, ρ) are pseudorectangles, the validity of $(ii) \Rightarrow (i)$ can be proved similarly if apply Theorem 5.6 instead of Theorem 5.3 and $Q \ \otimes_3 \ Q$ instead of $Q \ \otimes_1 \ Q$. ☐

In the next proposition, we denote by $\mathfrak{c}$ the cardinality of the continuum.

Proposition 5.10. *Let X be a nonempty set, let $\equiv$ be an equivalence relation on X and*

$$Q = \{X_j : j \in J\}$$

be the partition of X *corresponding to* $\equiv$. *Then the following statements hold*:

(*i*) *The inequality* $|J| \leqslant \mathfrak{c}$ *holds if and only if there is a strongly rigid pseudometric space* (X, d) *such that*

$$(x \equiv y) \Leftrightarrow (d(x, y) = 0) \tag{5.35}$$

is valid for all $x, y \in X$.

(*ii*) *The equality* $|J| = 4$ *holds if and only if there is a pseudorectangle* (X, d) *such that* (5.35) *is valid for all* $x, y \in X$.

Proof. Statement (*i*) was proved in Theorem 4.13 of [7]. Let us prove the validity of (*ii*).

Let $|J| = 4$ hold. Let us consider an injective mapping $f : Q \otimes_3 Q \to \mathbb{R}$ such that

$$f\left(\bigcup_{j\in J} X_j^2\right) = 0 \tag{5.36}$$

and

$$f(B) \in (1, 2), \tag{5.37}$$

whenever B is a block of $Q \otimes_3 Q$ defined by equalities (5.15)–(5.17).

Write $\equiv_3$ for the equivalence relation corresponding to the partition $Q \otimes_3 Q$ and denote by d the mapping

$$X^2 \overset{\pi\otimes\pi}{\to} Q \otimes_3 Q \xrightarrow{f} \mathbb{R}, \quad d \coloneqq f(\pi \otimes \pi),$$

where π and $\pi \otimes \pi$ are the canonical projections of X on $X/{\equiv}$ and X^2 on $X^2/{\equiv_3}$, respectively. By Theorem 5.3, (X, d) is a pseudorectangle. (We note only that (5.37) implies the triangle inequality for d.) The definition of d and Lemma 5.1 imply that Q is the partition of X corresponding the equivalence relation $\overset{0(d)}{=}$.

Let us consider a pseudorectangle (X, d) such that $Q = \{X_j : j \in J\}$ is a partition corresponding to the equivalence relation $\overset{0(d)}{=}$. Then the equality $|J| = 4$ follows from Theorem 5.6. □

Remark 5.11. By Proposition 5.9, all strongly rigid pseudometric spaces (pseudorectangles) (X, d) satisfying (5.35) are combinatorially similar. Thus, Proposition 5.10 can be considered as a development of Proposition 3.7.

The following lemma was proposed by anonymous reviewer of the present article.

Lemma 5.12. *Let* $\mathcal{CL}^*_{\text{met}}$ *be a class of nonempty metric spaces such that, for every* $(X, d) \in \mathcal{CL}^*_{\text{met}}$ *and each metric space* (Y, ρ), *we have*:

(*i*) $(Y, \rho) \in \mathcal{CL}^*_{\text{met}}$ *whenever* (X, d) *and* (Y, ρ) *are isometric.*

(*ii*) *If* $(Y, \rho) \in \mathcal{CL}^*_{\text{met}}$ *and* $Y = X$, *then the identical mapping* $\mathrm{Id} : X \to X$ *is a combinatorial similarity of* (X, d) *and* (Y, ρ).

(*iii*) *Every nonempty subspace of* (X, d) *belongs to* $\mathcal{CL}^*_{\text{met}}$.

Then, for all $(A, \alpha) \in \mathcal{CL}^*_{\text{met}}$ *and* $(B, \beta) \in \mathcal{CL}^*_{\text{met}}$, *the space* (B, β) *contains a subspace combinatorially similar to the space* (A, α) *whenever*

$$|A| \leqslant |B|. \tag{5.38}$$

Proof. Let (A, α) and (B, β) belong to $\mathcal{CL}^*_{\text{met}}$ and let (5.38) hold. We must show that (B, β) contains a subspace which is combinatorially similar to (A, α).

Let us consider first the case when

$$A \cap B = \varnothing. \tag{5.39}$$

Inequality (5.38) implies the existence of an injective mapping $F : A \to B$. Write

$$B_1 \coloneqq (B \setminus F(A)) \cup A \tag{5.40}$$

and define the mapping $\Phi : B_1 \to B$ by

$$\Phi(x) = \begin{cases} F(x) & \text{if } x \in A \\ x & \text{if } x \in B \setminus F(A). \end{cases}$$

From equalities (5.39) and (5.40) and the injectivity of F it follows that Φ is a bijection. Let us define a metric $\beta_1 : B_1^2 \to \mathbb{R}$ as follows:

$$\beta_1(x, y) = \beta(\Phi(x), \Phi(y)), \quad x, y \in B_1.$$

Then the spaces (B_1, β_1) and (B, β) are isometric and, consequently, $(B_1, \beta_1) \in \mathcal{CL}^*_{\mathrm{met}}$ holds by condition (i).

Let us denote by α_1 the restriction of the metric β_1 on the set A. Then, by using condition (iii), we see that

$$(A, \alpha_1) \in \mathcal{CL}^*_{\mathrm{met}}. \tag{5.41}$$

The last membership relation and condition (ii) imply that (A, α) and (A, α_1) are combinatorially similar. Thus, (B_1, β_1) contains a subspace which is combinatorially similar to (A, α). It follows from definition of the metric β_1 that $\Phi : B_1 \to B$ is an isometry of (B_1, β_1) and (B, β). Hence, the space (B, β) also contains the desired subspace. Thus, the conclusion of the lemma is valid if (5.39) is satisfied.

To complete the proof, we note that, for every $(B, \beta) \in \mathcal{CL}^*_{\mathrm{met}}$, there is a metric space (B_2, β_2) such that $B \cap B_2 = \varnothing$ and (B_2, β_2) is isometric to (B, β). Then, by condition (i), we have $(B_2, \beta_2) \in \mathcal{CL}^*_{\mathrm{met}}$ and, in addition, $|B_2| = |B| \geqslant |A|$. According to the aforementioned arguments, the space (B_2, β_2) contains a subspace, which is combinatorially similar to (A, α). Now the conclusion of the lemma follows from the isometricity of (B_2, β_2) and (B, β). □

The following theorem will use the notation:

- $\mathcal{CL}_{\mathrm{st}}$ for the class of all strongly rigid pseudometric spaces;
- $\mathcal{CL}_{di}$ for the class of all discrete pseudometric spaces;
- $\mathcal{CL}_{pr}$ for the class of all pseudorectangles.

Moreover, we denote by $\mathcal{CL}^4_{\mathrm{st}}$ the class of all $(X, d) \in \mathcal{CL}_{\mathrm{st}}$ satisfying the inequality $|d(X^2)| \leqslant 4$.

Theorem 5.13. *Let $\mathcal{CL}$ be a class of nonempty pseudometric spaces such that for every $(X, d) \in \mathcal{CL}$ and each pseudometric space (Y, ρ) we have*:

(i_1) *$(Y, \rho) \in \mathcal{CL}$ whenever (X, d) and (Y, ρ) are pseudoisometric.*

(i_2) *If $(Y, \rho) \in \mathcal{CL}$, and $Y = X$, and the relations $\overset{0(d)}{=}$ and $\overset{0(\rho)}{=}$ are the same, then the identical mapping $Id_X : X \to X$ is a combinatorial similarity of (X, d) and (Y, ρ).*

(i_3) *Every nonempty subspace of (X, d) belongs to $\mathcal{CL}$.*

Then at least one from the inclusions

$$\mathcal{CL} \subseteq \mathcal{CL}_{\mathrm{st}}, \quad \mathcal{CL} \subseteq \mathcal{CL}_{di}, \quad \mathcal{CL} \subseteq \mathcal{CL}_{pr} \cup \mathcal{CL}^4_{\mathrm{st}} \tag{5.42}$$

holds. Moreover, if $\mathcal{CL}$ is maximal, then it coincides with $\mathcal{CL}_{\mathrm{st}}$, or $\mathcal{CL}_{di}$, or $\mathcal{CL}_{pr} \cup \mathcal{CL}^4_{\mathrm{st}}$.

Remark 5.14. The maximality of $\mathcal{CL}$ means that for every class $\mathcal{CL}^o$ of nonempty pseudometric spaces, the inclusion $\mathcal{CL}^o \supseteq \mathcal{CL}$ implies the equality $\mathcal{CL}^o = \mathcal{CL}$ whenever $\mathcal{CL}^o$ satisfies conditions $(i_1) - (i_3)$ for every $(X, d) \in \mathcal{CL}^o$ and every pseudometric space (Y, ρ).

Proof of Theorem 5.13. First, we prove the equality

$$\mathbf{Cs}(X/\overset{0}{=}, \delta_d) = \mathrm{Sym}(X/\overset{0}{=}) \tag{5.43}$$

for every $(X, d) \in \mathcal{CL}$.

Let us consider an arbitrary $(X, d) \in \mathcal{CL}$, and let $(X/\overset{0}{=}, \delta_d)$ be the metric reflection of (X, d). It follows directly from Definition 2.5 and Proposition 2.9 that the canonical projection $\pi : X \to X/\overset{0}{=}$ is a pseudoisometry of (X, d) and $(X/\overset{0}{=}, \delta_d)$. Hence, $(X/\overset{0}{=}, \delta_d)$ belongs to $\mathcal{CL}$ by condition (i_1). Since $(X/\overset{0}{=}, \delta_d)$ is a metric space, $\overset{0(\delta_d)}{=}$ is the identical relation on $X/\overset{0}{=}$, i.e., for all $a, b \in X/\overset{0}{=}$, $\langle a, b\rangle \in \overset{0(\delta_d)}{=}$ holds if and only if $a = b$.

Let $\Phi : X/\overset{0}{=} \to X/\overset{0}{=}$ be an arbitrary bijection of $X/\overset{0}{=}$. The function $\rho^{\Phi} : (X/\overset{0}{=})^2 \to \mathbb{R}$ defined as

$$\rho^{\Phi}(a, b) = \delta_d(\Phi(a), \Phi(b)) \tag{5.44}$$

is a metric on $X/\overset{0}{=}$ and Φ is an isometry of the metric spaces $(X/\overset{0}{=}, \delta_d)$ and $(X/\overset{0}{=}, \rho^{\Phi})$. Since every isometry is a pseudoisometry, the membership $(X/\overset{0}{=}, \delta_d) \in \mathcal{CL}$ implies $(X/\overset{0}{=}, \rho^{\Phi}) \in \mathcal{CL}$ by condition (i_1). Furthermore, we have

$$(\langle a, b\rangle \in \overset{0(\rho)}{=}) \Leftrightarrow (\rho^{\Phi}(a, b) = 0)$$

for all $a, b \in X/\overset{0}{=}$, because ρ is a metric on $X/\overset{0}{=}$. Thus, the relations $\overset{0(\delta_d)}{=}$ and $\overset{0(\rho^{\Phi})}{=}$ are the same. Consequently, by condition (i_2), the identical mapping of $X/\overset{0}{=}$ is a combinatorial similarity of $(X/\overset{0}{=}, \delta_d)$ and $(X/\overset{0}{=}, \rho^{\Phi})$. Now using (5.44) and Definition 2.2, we obtain that Φ is a combinatorial self-similarity of $(X/\overset{0}{=}, \delta_d)$. Since Φ is an arbitrary permutation of $X/\overset{0}{=}$, equality (5.43) holds.

By Theorem 4.5, equality (5.43) implies the inclusion

$$\mathcal{CL} \subseteq \mathcal{CL}_{\mathrm{st}} \cup \mathcal{CL}_{di} \cup \mathcal{CL}_{pr}. \tag{5.45}$$

Our next goal is to prove that at least one of inclusions (5.42) holds. Let us denote by $\mathcal{CL}^*_{\mathrm{met}}$ the subclass of all metric spaces which belong $\mathcal{CL}$. Then, by using (5.45) and the definitions of the pseudorectangles, strongly rigid spaces and discrete spaces, we see that at least one of inclusions (5.42) holds if and only if we have at least one of the inclusions

$$\mathcal{CL}^*_{\mathrm{met}} \subseteq \mathcal{CL}_{\mathrm{st}}, \quad \mathcal{CL}^*_{\mathrm{met}} \subseteq \mathcal{CL}_{di}, \quad \mathcal{CL}^*_{\mathrm{met}} \subseteq \mathcal{CL}_{pr} \cup \mathcal{CL}^4_{\mathrm{st}}. \tag{5.46}$$

It is clear that all inclusions in (5.46) are simultaneously valid, if $|X| \leqslant 2$ holds for every $(X, d) \in \mathcal{CL}^*_{\mathrm{met}}$.

To analyze the case when $|X| \geqslant 3$ holds for some $(X, d) \in \mathcal{CL}^*_{\mathrm{met}}$, we will use Lemma 5.12.

First, we note that the definition of $\mathcal{CL}$ implies that $\mathcal{CL}^*_{\mathrm{met}}$ satisfies conditions (i)–(iii) of Lemma 5.12 for every $(X, d) \in \mathcal{CL}^*_{\mathrm{met}}$ and each metric space (Y, ρ).

Let us consider the special case when

$$|Y| \leqslant 3 \tag{5.47}$$

holds for every $(Y, \rho) \in \mathcal{CL}^*_{\mathrm{met}}$, and there exists $(X, d) \in \mathcal{CL}^*_{\mathrm{met}}$ with

$$|X| = 3. \tag{5.48}$$

Then, by using (5.45) and the inclusion $\mathcal{CL}^*_{\mathrm{met}} \subseteq \mathcal{CL}$, we obtain that (X, d) is either strongly rigid or discrete. Suppose that (X, d) is strongly rigid. By Lemma 5.12, inequality (5.47) and equality (5.48) imply that (Y, ρ) is combinatorially similar to subspace of (X, d). Consequently, (Y, ρ) is strongly rigid because combinatorial similarities preserve the strong rigidity and every subspace of strongly rigid space is also strongly rigid. Thus,

$$\mathcal{CL}^*_{\mathrm{met}} \subseteq \mathcal{CL}_{\mathrm{st}} \tag{5.49}$$

holds if (X, d) is strongly rigid. Arguing similarly, we obtain that

$$\mathcal{CL}^*_{\mathrm{met}} \subseteq \mathcal{CL}_{di} \tag{5.50}$$

holds if (X, d) is discrete.

To complete the proof that on of inclusions (5.46) is fulfilled, it is necessary to consider the case when there is $(X, d) \in \mathcal{CL}^*_{\mathrm{met}}$ such that

$$|X| \geqslant 4. \tag{5.51}$$

To do it, we note that (5.45) implies $(X, d) \in \mathcal{CL}_{\mathrm{st}}$, or $(X, d) \in \mathcal{CL}_{di}$, or $(X, d) \in \mathcal{CL}_{pr}$. If $(X, d) \in \mathcal{CL}_{\mathrm{st}}$, then, using Lemma 5.12, we obtain that, for every $(Y, \rho) \in \mathcal{CL}^*_{\mathrm{met}}$, each subspace of (Y, ρ) is strongly rigid whenever this subspace contains at most four points. The last statement implies $(Y, \rho) \in \mathcal{CL}_{\mathrm{st}}$ by definition of strongly rigid metric spaces (see (2.7)–(2.8)). Thus, inclusion (5.49) holds if we have $(X, d) \in \mathcal{CL}^*_{\mathrm{met}} \cap \mathcal{CL}_{\mathrm{st}}$ satisfying (5.51).

Arguing in a similar way, it can be proven that $(X, d) \in \mathcal{CL}^*_{\mathrm{met}} \cap \mathcal{CL}_{di}$ and (5.51) imply (5.50).

Suppose now that

$$(X, d) \in \mathcal{CL}_{pr} \tag{5.52}$$

and (5.51) holds, and consider an arbitrary (Y, ρ) belonging to $\mathcal{CL}^*_{\mathrm{met}}$. We claim that

$$(Y, \rho) \in \mathcal{CL}_{pr} \cup \mathcal{CL}^4_{\mathrm{st}}. \tag{5.53}$$

To prove (5.53), note that (5.52) implies the inequality

$$|Y| \leqslant 4. \tag{5.54}$$

Indeed, if $|Y| \geqslant 5$ holds, then it follows from (5.45) that

$$(Y, \rho) \in \mathcal{CL}_{\mathrm{st}} \quad \text{or} \quad (Y, \rho) \in \mathcal{CL}_{di}.$$

Now, by using $|Y| \geqslant 4$ instead of (5.51), as in the proofs of (5.49)–(5.50), we obtain

$$(X, d) \in \mathcal{CL}_{\mathrm{st}} \quad \text{or} \quad (X, d) \in \mathcal{CL}_{di},$$

contrary to (5.52). Inequality (5.54) follows.

Since (5.52) is satisfied only under condition $|X| = 4$, the inequality (5.54) implies that (Y, ρ) is combinatorially similar to a subspace of (X, d) by Lemma (5.12). It follows directly from the definition of $\mathcal{CL}^4_{\mathrm{st}}$, Definition (2.20) and (5.52) that all subspaces of (X, d) belong to $\mathcal{CL}_{pr} \cup \mathcal{CL}^4_{\mathrm{st}}$. Hence, (5.53) holds.

Let $\mathcal{CL}$ be maximal in the sense of Remark 5.14. Our final goal is to prove that one of the following equalities

$$\mathcal{CL} = \mathcal{CL}_{\mathrm{st}}, \quad \mathcal{CL} = \mathcal{CL}_{di}, \quad \mathcal{CL} = \mathcal{CL}_{pr} \cup \mathcal{CL}^4_{\mathrm{st}} \tag{5.55}$$

holds. To prove it, we note that the classes $\mathcal{CL}_{\mathrm{st}}$, $\mathcal{CL}_{di}$ and $\mathcal{CL}_{pr} \cup \mathcal{CL}^4_{\mathrm{st}}$ satisfy conditions (i_1) − (i_3) of the theorem for each pseudometric space (Y, ρ) and every (X, d) belonging to these classes. Now rewriting (5.42) as

$$\mathcal{CL}_{\mathrm{st}} \supseteq \mathcal{CL}, \quad \mathcal{CL}_{di} \supseteq \mathcal{CL}, \quad \mathcal{CL}_{pr} \cup \mathcal{CL}^4_{\mathrm{st}} \supseteq \mathcal{CL} \tag{5.56}$$

and, using the maximality of $\mathcal{CL}$, we see that each of inclusions (5.56) implies the corresponding equality in (5.55). ☐

Corollary 5.15. *Let (Z, l) be a nonempty pseudometric space. Then the equality*

$$\mathbf{Cs}(Z/\overset{0}{=}, \delta_l) = \mathrm{Sym}(Z/\overset{0}{=})$$

holds, if and only if there is a class $\mathcal{CL}$ of nonempty pseudometric spaces, which satisfies $(Z, l) \in \mathcal{CL}$ and conditions (i_1)–(i_3) of Theorem 5.13 *for every $(X, d) \in \mathcal{CL}$ and each nonempty pseudometric space (Y, ρ).*

In connection with Theorem 5.13, the following question naturally arises. Are conditions (i_1) − (i_3) independent of each other?

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References

[1] V. Bilet and O. Dovgoshey, *Completeness, closedness and metric reflections of pseudometric spaces*, Topology Appl. **327** (2023), Article ID 108440, 14 p.

[2] V. Bilet and O. Dovgoshey, *When all permutations are combinatorial similarities*, Bull. Korean Math. Soc. **60** (2023), no. 3, 733–746.

[3] V. Bilet, O. Dovgoshey, M. Küçükaslan, and E. Petrov, *Minimal universal metric spaces*, Ann. Acad. Sci. Fenn. Math. **42** (2017), no. 2, 1019–1064.

[4] K. A. Broughan, *A metric characterizing Čech dimension zero*, Proc. Amer. Math. Soc. **39** (1973), 437–440.

[5] O. Dovgoshey, *Semigroups generated by partitions*, Int. Electron. J. Algebra **26** (2019), 145–190.

[6] O. Dovgoshey, *Combinatorial properties of ultrametrics and generalized ultrametrics*, Bull. Belg. Math. Soc. Simon Stevin **27** (2020), no. 3, 379–417.

[7] O. Dovgoshey and J. Luukkainen, *Combinatorial characterization of pseudometrics*, Acta Math. Hungar. **161** (2020), no. 1, 257–291.

[8] O. Dovgoshey and R. Shanin, *Uniqueness of best proximity pairs and rigidity of semimetric spaces*, J. Fixed Point Theory Appl. **25** (2023), no. 1, Paper No 34, 31 p.

[9] Y. Hattori, *Congruence and dimension of nonseparable metric spaces*, Proc. Amer. Math. Soc. **108** (1990), no. 4, 1103–1105.

[10] Y. Ishiki, *Strongly rigid metrics in spaces of metrics*, Topology Proceedings **63** (2024), 125–148.

[11] L. Janos, *A metric characterization of zero-dimensional spaces*, Proc. Amer. Math. Soc. **31** (1972), no. 1, 268–270.

[12] L. Janos and H. Martin, *Metric characterizations of dimension for separable metric spaces*, Proc. Amer. Math. Soc. **70** (1978), no. 2, 209–212.

[13] J. L. Kelley, *General Topology*, Springer-Verlag, New York, Heidelberg, Berlin, 1975.

[14] K. Kuratowski and A. Mostowski, *Set Theory with an Introduction to Descriptive Set Theory*, North-Holland Publishing Company, Amsterdam, New York, Oxford, 1976.

[15] Đ. Kurepa, *Tableaux ramifiés daensemples, espaces pseudodistacies*, C. R. Acad. Sci. Paris **198** (1934), 1563–1565.

[16] H. W. Martin, *Strongly rigid metrics and zero dimensionality*, Proc. Amer. Math. Soc. **67** (1977), no. 1, 157–161.

[17] O. Ore, *Theory of equivalence relations*, Duke Math. J. **9** (1942), no. 3, 573–627.

[18] J. Rouyer, *Generic properties of compact metric spaces*, Topology Appl. **158** (2011), no. 16, 2140–2147.

[19] M. O. Searcóid, *Metric Spaces*, Springer, Verlag, London, 2007.