Location in circulant graphs

Ville Junnila and **Tero Laihonen**
Department of Mathematics and Statistics
University of Turku, FI-20014 Turku, Finland
viljun@utu.fi and terolai@utu.fi

Gabrielle Paris*
LIRIS, University of Lyon, France
claudia.paris-sierra@univ-lyon1.fr

1 Introduction

Let $G = (V, E)$ be a simple, undirected graph with the vertex set V and the edge set E. Let $N[u]$ denote the *closed neighbourhood* of a vertex $u \in V$. A nonempty subset $C \subseteq V$ is called a *code*, and its elements are called *codewords*. The *I-set* (or the *identifier*) of u is defined as

$$I(u) = I(G, C; u) = N[u] \cap C.$$

Let C be a code in G. A vertex $u \in V$ is *covered* or *dominated* by a codeword of C if the identifying set $I(u)$ is nonempty. The code C is *dominating* in G if all the vertices of V is covered by a codeword of C, i.e., $|I(u)| \geq 1$ for all $u \in V$. The code C is *identifying* in G if C is dominating and for all distinct $u, v \in V$ we have

$$I(u) \neq I(v).$$

The definition of identifying codes is due to Karpovsky *et al.* [14], and the original motivation for studying such codes comes from fault diagnosis in multiprocessor systems. The concept of locating-dominating codes is closely related to the one of identifying codes. We say that the code is *locating-dominating* in G if C is dominating and for all distinct $u, v \in V \setminus C$ we have $I(u) \neq I(v)$. The definition of locating-dominating codes was introduced by Slater [17, 19, 20]. For the extensive literature on these codes consult [15]. A code $C \subseteq V$ is *self-identifying* in $G = (V, E)$ if for all vertices $u \in V$ we have

$$\bigcap_{c \in I(u)} N[c] = \{u\}.$$

Self-identifying codes are discussed in [9, 11, 12]. An identifying, locating-dominating or self-identifying code with the smallest cardinality in a given finite graph G is called *optimal*. The number of codewords in an optimal identifying and locating-dominating code in a finite graph G is denoted by $\gamma^{ID}(G)$, $\gamma^{LD}(G)$ and $\gamma^{SID}(G)$, respectively.

In this paper, we focus on studying these codes in so called circulant graphs. Let n and $d_1, \ldots, d_k$ be positive integers such that for all i, $d_i \leq \frac{n}{2}$. The circulant graph $C_n(d_1, d_2, \ldots, d_k)$ is defined as follows: the vertex set is $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and the neighbourhood of a vertex $u \in \mathbb{Z}_n$ is

$$N[u] = \{u, u \pm d_1, u \pm d_2, \ldots, u \pm d_k\},$$

where the calculations are done modulo n. Previously, in [1, 3, 5, 7, 10, 16, 18, 21], identifying and locating-dominating codes have been studied in the circulant graphs $C_n(1, 2, \ldots, r)$ $(r \in \mathbb{Z}, r \geq 1)$,

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which can also be viewed as power graphs of cycles of length n. Recently, in [6], Ghebleh and Niepel studied identification and location-domination in $C_n(1,3)$. They obtained the following results:

$$\lceil 4n/11 \rceil \le \gamma^{ID}(C_n(1,3)) \le \lceil 4n/11 \rceil + 1 \quad \text{and} \quad \lceil n/3 \rceil \le \gamma^{LD}(C_n(1,3)) \le \lceil n/3 \rceil + 1.$$

In this paper, we give lower bounds on values of $\gamma^{LD}(G)$ and $\gamma^{SID}(G)$ for the circulant graphs $C_n(1, d-1, d, d+1)$ with $d > 3$ and give code constructions attaining these bounds for infinitely many values of parameters n and d. Moreover, we a give lower bound on $\gamma^{ID}(C_n(1, d-1, d, d+1))$ and we construct a family of identifying codes whose cardinality approaches the lower bound as n tends to infinity. In addition, we give optimal values for self-identifying codes in the circulant graph $C_n(1, n/2)$ when n is even.

2 Lower bounds using an infinite grid

Let us first present the definition of the *infinite king grid* $\mathcal{K}$. Its vertex set is $V = \mathbb{Z}^2$. The edges of the infinite king grid $\mathcal{K}$ are defined in such a way that the closed neighbourhood of $u = (x, y) \in \mathbb{Z}^2$ is

$$N[u] = \{(x', y') \in \mathbb{Z}^2 \mid \max\{|x - x'|, |y - y'|\} \le 1\}.$$

For estimating the sizes of infinite codes, we need a way to measure them in the grid. For this purpose, we first denote

$$Q_m = \{(x, y) \in \mathbb{Z}^2 \mid |x| \le m, |y| \le m\},$$

where m is a positive integer. The *density* of a code $C \subseteq \mathbb{Z}^2$ is then defined as

$$D(C) = \limsup_{m \to \infty} \frac{|C \cap Q_m|}{|Q_m|}.$$

For a finite nonempty set $S \subseteq V$ in a graph $G = (V, E)$, the (local) *density* of a code $C \subseteq V$ in S is defined as $|S \cap C|/|S|$.

Analogously to finite graphs, an identifying, locating-dominating and self-identifying code with the smallest possible density in the infinite king grid is called *optimal*. The densities of optimal codes have been intensively studied and all the exact values are known. The optimal densities can be found in Table 1 together with the references to the papers, where the results have been presented.

	density	reference
ID	2/9	[2, 4]
LD	1/5	[8]
self-ID	1/3	[9]

Table 1: The densities of optimal identifying (ID), locating-dominating (LD) and self-identifying (self-ID) codes in the king grid $\mathcal{K}$.

In the following theorem, we present the connection between identifying, locating-dominating and self-identifying codes in the infinite king grid and the circulant graphs $C_n(1, d-1, d, d+1)$.

Theorem 1. *Let n, d and k be positive integers such that $d \ge 3$. If C is an identifying code in $C_n(1, d-1, d, d+1)$ with k codewords, then there exists an identifying code in the infinite king grid $\mathcal{K}$ with density k/n. Analogous results also hold for locating-dominating and self-identifying codes.*

In the following corollary, we give lower bounds for codes in the circulant graphs $C_n(1, d-1, d, d+1)$. In Section 3, we show that the lower bounds can be attained with locating-dominating and self-identifying codes and that there exists an infinite family of identifying codes approaching the lower bound.

Corollary 2. *Let n and d be positive integers such that $d \geq 3$ and $G = C_n(1, d-1, d, d+1)$. Then we have*

$$\gamma^{LD}(G) \geq \left\lceil \frac{n}{5} \right\rceil, \gamma^{ID}(G) \geq \left\lceil \frac{2n}{9} \right\rceil \text{ and } \gamma^{SID}(G) \geq \left\lceil \frac{n}{3} \right\rceil.$$

3 Optimal constructions

In the following theorem, we give optimal locating-dominating codes in the circulant graph $C_n(1, d-1, d, d+1)$. Furthermore, we give an infinite sequence of identifying codes approaching the lower bound in Corollary 2.

Theorem 3. *(i) For $d \equiv 8 \pmod{10}$, $d \geq 8$, $n \geq 4d+6$ and $n \equiv 0 \pmod{10}$, we have*

$$\gamma^{LD}(C_n(1, d-1, d, d+1)) = \frac{n}{5}.$$

(ii) There is a sequence of identifying codes $(C_k)_{k=1}^{\infty}$ in the circulant graphs $C_n(1, d-1, d, d+1)$ with

$$\lim_{k \to \infty} \frac{|C_k|}{n} = \frac{2}{9}.$$

In the next theorem, we will show that the bound on self-identifying codes in Corollary 2 can be reached.

Theorem 4. *If $d \equiv 1 \pmod 3$, $n \geq 3d+5$ and $n \equiv 0 \pmod 3$, then*

$$\gamma^{SID}(C_n(1, d-1, d, d+1)) = \frac{n}{3}.$$

In the following theorem, we give optimal self-identifying codes for $C_n(1, n/2)$ for n even.

Proposition 5. *Let $k \geq 5$. The optimal cardinality of self-identifying code in $C_{2k}(1, k)$ is as follows:*

$$\gamma^{SID}(C_{2k}(1,k)) = \begin{cases} \left\lceil 4\frac{k}{3} \right\rceil & \text{if } k \equiv 0 \pmod 3 \text{ or } k \equiv 1 \pmod 3 \\ \left\lceil 4\frac{k}{3} \right\rceil + 1 & \text{if } k \equiv 2 \pmod 3 \end{cases}.$$

It is natural to study also other circulant graphs. For optimal codes in circulant graphs $C_n(1, d)$ and $C_n(1, d-1, d)$, see [13].

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