Mathematische Annalen
https://doi.org/10.1007/s00208-023-02667-x

Mathematische Annalen

A note on exceptional characters and non-vanishing of Dirichlet L-functions

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Received: 15 March 2023 / Revised: 22 June 2023 / Accepted: 26 June 2023

Abstract
We study non-vanishing of Dirichlet L-functions at the central point under the unlikely assumption that there exists an exceptional Dirichlet character. In particular we prove that if ψ is a real primitive character modulo $D \in \mathbb{N}$ with $L(1, \psi) \ll (\log D)^{-25-\varepsilon}$, then, for any prime $q \in [D^{300}, D^{O(1)}]$, one has $L(1/2, \chi) \neq 0$ for almost all Dirichlet characters $\chi \pmod q$.

Mathematics Subject Classification 11M20

1 Introduction

A central problem in analytic number theory is the study of vanishing or non-vanishing of L-functions at the central point. Some arithmetic consequences arise for example due to the Birch and Swinnerton-Dyer conjecture, which links the order of the central zero of an elliptic curve L-function with its rank (see for example [13]). Another application was provided by Iwaniec and Sarnak [7], who proved that at least 50% of L-functions in certain families of cusp forms do not vanish at the central point, and showed that any improvement on this proportion would rule out the existence of Landau-Siegel zeros.

A conjecture attributed to Chowla states that $L(1/2, \chi) \neq 0$ for all Dirichlet characters χ (see [3] for the conjecture in the case of real Dirichlet characters). In this paper we study the non-vanishing of Dirichlet L-functions at the central point under the unlikely assumption that there exists an exceptional Dirichlet character.

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Unconditionally, Balasubramanian and Murty [1] were the first to show that for any sufficiently large prime q, one has that $L(1/2, \chi) \neq 0$ for a positive proportion of Dirichlet characters $\chi \pmod q$. This result was significantly improved by Iwaniec and Sarnak [5], who obtained the non-vanishing proportion $1/3 - \varepsilon$ (also for non-prime q). The best known result for prime moduli is $5/13 - \varepsilon$ due to Khan, Milićević, and Ngo [8].

The works of Murty [11] and Bui et al. [2] prove the non-vanishing proportion $1/2 - o(1)$ conditionally—Murty under the generalized Riemann hypothsesis, and Bui, Pratt, and Zaharescu under the existence of an exceptional character with modulus of suitable size.

In this paper we improve on the result of Bui, Pratt, and Zaharescu. In particular we obtain the following corollary, improving their proportion $1/2 - o(1)$ to $1 - o(1)$.

Corollary 1 *Let $\varepsilon > 0$ be fixed. Let $D > 1$ be a squarefree fundamental discriminant and let ψ be the associated primitive quadratic character modulo D. Assume that*

$$L(1, \psi) \ll \frac{1}{(\log D)^{25+\varepsilon}}. \tag{1}$$

Then, for any fixed $C > 300$ and any prime q such that

$$D^{300} \leq q \leq D^C,$$

we have

$$|\{\chi \pmod q \colon L(1/2, \chi) \neq 0\}| = (1 + o(1))\varphi(q),$$

where the rate of convergence of $o(1)$ depends only on ε, C and the implied constant in (1).

Remark 1 It is feasible that it is possible to loosen the condition (1) to the condition that $L(1, \psi) = o(1/\log D)$. This would require reworking the arguments in [2] with a more optimally chosen mollifier than (5) below, and being very careful about not losing any logarithmic factors. It might be possible to carry this out by adapting the arguments from [4]. In [4] Conrey and Iwaniec considered a related problem, showing that if $L(1, \psi) = o(1/\log D)$, then, for any Dirichlet L-function, almost all zeros, whose imaginary part is on a suitable range, are simple and lie on the critical line.

As in [2], we actually get a more quantitative result — the following holds unconditionally, but is non-trivial only in case an exceptional character exists.

Theorem 2 *Let $\varepsilon > 0$ be fixed. Let $D > 1$ be a squarefree fundamental discriminant and let ψ be the associated primitive quadratic character modulo D. Let $C > 300$ be fixed and let q be a prime such that*

$$D^{300} \leq q \leq D^C. \tag{2}$$

Then, for any $\delta > 0$,

$$\frac{1}{\varphi(q)} \sum_{\chi \pmod q} \mathbf{1}_{|L(1/2,\chi)| \geq \frac{\delta^{3/2}}{(\log q)^{9/2}}} = 1 + O\left(\delta^{-2} L(1,\psi)(\log q)^{25+\varepsilon} + \frac{\delta^{-2}}{(\log q)^{1-\varepsilon}} + \delta\right). \tag{3}$$

Corollary 1 immediately follows from applying Theorem 2 with $\delta = (\log q)^{-\varepsilon/4}$ and $\varepsilon/3$ in place of ε. In Theorem 2 and other statements, the implied constants are allowed to depend on ε and C (which are said to be fixed), but not on D or q.

Remark 3 We have not tried to optimize the lower bound we get for $|L(1/2, \chi)|$ in Theorem 2. By estimating the left hand side of (14) below more carefully, it would probably be possible to improve the power of $\log q$ in the lower bound. Furthermore, similarly to Remark 1, it might be possible to improve on the error term.

As in many works, we consider only the even primitive characters, the case of odd primitive characters being handled similarly (since q is prime, there is only one non-primitive character χ_0 so its contribution is negligible). We write $\sum^{+}$ for a sum over primitive even characters modulo q, and $\varphi^{+}(q)$ for the number of such characters.

Our proof is based on the work of Bui, Pratt and Zaharescu [2] and the equidistribution of the product $\varepsilon(\chi)\varepsilon(\psi\chi)$ of root numbers. Here and later, $\varepsilon(\chi)$ denotes the sign of the functional equation of $L(s, \chi)$, which can also be written as a normalized Gauss sum

$$\varepsilon(\chi) := \frac{\tau(\chi)}{q^{1/2}} = \frac{1}{q^{1/2}} \sum_{\substack{a \pmod q \\ (a,q)=1}} \chi(a) e\left(\frac{a}{q}\right). \tag{4}$$

The following proposition which we will prove in Sect. 4 shows that $\varepsilon(\chi)\varepsilon(\psi\chi)$ is equidistributed on the unit circle when χ runs over even primitive characters modulo q.

Proposition 4 *Let $D > 1$ be a squarefree fundamental discriminant and let ψ be the associated primitive quadratic character modulo D. Let q be a prime such that $q \nmid D$.*

For each character $\chi \pmod q$, let $\theta_\chi \in (0, 2\pi]$ be such that $\varepsilon(\chi)\varepsilon(\chi\psi) = e(\theta_\chi)$. Then, for any interval $(\alpha, \beta] \subseteq (0, 2\pi]$, we have

$$\frac{1}{\varphi^{+}(q)} \sum_{\chi \pmod q}^{+} \mathbf{1}_{\theta_\chi \in (\alpha,\beta]} = \beta - \alpha + O(q^{-1/4}),$$

where the implied constant is absolute.

2 The work of Bui, Pratt and Zaharescu

Let ε, D, ψ, C, and q be as in Theorem 2. Following [2], for any character $\chi \pmod q$, we write

$$L_\chi(s) := L(s,\chi)L(s,\chi\psi) = \sum_{n\geq 1} \frac{(1*\psi)(n)\chi(n)}{n^s}.$$

Note that Theorem 2 is non-trivial only when ψ is an exceptional character modulo D (in the sense that (1) holds for some $\varepsilon > 0$), and in this case we expect $1*\psi(n)$ to vanish often once $n > D^2$, say (see for example [2, formula (2.2)]).

Bui, Pratt and Zaharescu [2] consider the mollified L-functions $L_\chi(1/2)M(\chi)$, where the mollifier is taken as

$$M(\chi) := \sum_{\substack{n\leq X,\\ D\nmid n}} \frac{(\mu * \mu\psi)(n)\chi(n)}{\sqrt{n}} \tag{5}$$

with $X := D^{20}$. In the classical setting, it is crucial to take the mollifier as long as possible to make many of the coefficients of the mollified L-function vanish; in the exceptional case, the coefficients $(1*\psi)(n)$ of $L_\chi(1/2)$ are lacunary once n is larger than a small power of D, so taking $X = q^\kappa$ for a small $\kappa > 0$ is sufficient. This explains why we can obtain better non-vanishing results assuming the existence of exceptional characters.

For convenience, we make the same choice of parameters as [2], so that $X = D^{20}$ and (2) holds.

Write $Q = q\sqrt{D}/\pi$. Then [2, formula (4.1)] yields

$$L_\chi(1/2)M(\chi) = V_1\left(\frac{1}{Q}\right) + \varepsilon(\chi)\varepsilon(\chi\psi)V_2\left(\frac{1}{Q}\right) + O\left(|B_1(\chi)| + |B_2(\chi)|\right), \tag{6}$$

where, for $j = 1, 2$, $V_j(x)$ is a smooth weight as in [2, Section 3] and

$$B_j(\chi) := \sum_{\substack{a\leq X,\\ D\nmid a,\\ an>1}} \frac{(\mu*\mu\psi)(a)(1*\psi)(n)\chi(an)}{\sqrt{an}} V_j\left(\frac{n}{Q}\right).$$

The formula (6) is obtained in [2] using the approximate functional equation (see [2, Lemma 3.2]) and isolating the first summand in each term. By [2, Lemma 3.4] we have $V_j(1/Q) = 1 + O(Q^{-1/2+\varepsilon})$ for $j = 1, 2$ and hence (6) implies that, for some absolute constant $C_0 \geq 1$,

$$\left|L_\chi(1/2)M(\chi) - (1+\varepsilon(\chi)\varepsilon(\chi\psi))\right| \leq C_0\left(|B_1(\chi)| + |B_2(\chi)| + Q^{-1/2+\varepsilon}\right). \tag{7}$$

Furthemore, [2, Proposition 4.1] gives that, for $j = 1, 2$, any $\varepsilon > 0$, and any prime q in the range (2),

$$\sideset{}{^+}\sum_{\chi \pmod q} \left|B_j(\chi)\right|^2 \ll L(1,\psi)q(\log q)^{25+\varepsilon} + \frac{q}{(\log q)^{1-\varepsilon}}. \tag{8}$$

The strategy of Bui, Pratt and Zaharescu [2] is to proceed with the usual method of applying the Cauchy-Schwarz inequality to obtain that

$$\sideset{}{^+}\sum_{\substack{\chi \pmod q \\ L(1/2,\chi)\neq 0}} 1 \geq \sideset{}{^+}\sum_{\substack{\chi \pmod q \\ L_\chi(1/2)M(\chi)\neq 0}} 1 \geq \frac{\left|\sideset{}{^+}\sum_{\chi \pmod q} L_\chi(1/2)M(\chi)\right|^2}{\sideset{}{^+}\sum_{\chi \pmod q} \left|L_\chi(1/2)M(\chi)\right|^2}. \tag{9}$$

Bui, Pratt and Zaharescu then use (6) and (8) to compute the first and second moments of $L_\chi(1/2)M(\chi)$ — this gives that (assuming that ψ is an exceptional character), on the right hand side of (9), the numerator equals $(1+o(1))\varphi^+(q)^2$ whereas the denominator equals $(2+o(1))\varphi^+(q)$. This yields the non-vanishing proportion $1/2 + o(1)$.

Note that the application of the Cauchy-Schwarz inequality in (9) is costly, because by (7) the mollified L-functions still oscillate — our strategy is to dispose of the use of the Cauchy-Schwarz inequality and instead exploit the fact that (7) holds for individual L-functions. Actually, from (7), the equidistribution of the signs $\varepsilon(\chi)\varepsilon(\chi\psi)$ (Proposition 4) and (8), one can see that $L_\chi(1/2)M(\chi)$ for even $\chi \pmod q$ are equidistributed in the circle $|z-1| = 1$, which directly implies the non-vanishing of $L_\chi(1/2)M(\chi)$ for almost all characters $\chi \pmod q$.

The variation of the root number has been utilized also in earlier (unconditional) results that involved a two-piece mollifier (see e.g. [9] and [8]) — in these works one used the Cauchy-Schwarz inequality, but optimized its application.

3 Proof of Theorem 2 assuming Proposition 4

In this section we prove Theorem 2 assuming Proposition 4. Let ε, D, ψ, C, q, and δ be as in Theorem 2 and let C_0 be as in (7). Now (8) implies that

$$\sideset{}{^+}\sum_{\chi \pmod q} \mathbf{1}_{|B_j(\chi)|\geq \delta/(4C_0)} \ll \delta^{-2}\left(L(1,\psi)q(\log q)^{25+\varepsilon/2} + \frac{q}{(\log q)^{1-\varepsilon/2}}\right).$$

Furthermore, by Proposition 4 we know that $|1 + \varepsilon(\chi)\varepsilon(\chi\psi)| \geq 2\delta$ for all $\chi \pmod q$ apart from an exceptional set consisting of $\ll \delta\varphi(q) + q^{3/4}$ characters.

Combining (7) with the triangle inequality and these observations we obtain that, apart from an exceptional set of size

$$\ll \delta^{-2}L(1,\psi)q(\log q)^{25+\varepsilon} + \delta^{-2}\frac{q}{(\log q)^{1-\varepsilon}} + \delta\varphi(q), \tag{10}$$

we have

$$\left|L_\chi(1/2)M(\chi)\right| \geq 2\delta - C_0\left(\frac{\delta}{4C_0}+\frac{\delta}{4C_0}+Q^{-1/2+\varepsilon}\right) > \delta. \tag{11}$$

This already yields (3) with $\mathbf{1}_{|L(1/2,\chi)|\geq\delta^{3/2}/(\log q)^{9/2}}$ replaced by $\mathbf{1}_{|L(1/2,\chi)|\neq 0}$ and is thus sufficient for obtaining Corollary 1. We next proceed to showing a lower bound for $|L(1/2,\chi)|$ outside an acceptable exceptional set.

Recall that

$$L_\chi(1/2)M(\chi) = L(1/2,\chi)L(1/2,\chi\psi)M(\chi).$$

Since (11) holds apart from an exceptional set of size (10), Theorem 2 follows if we can establish that there are at most $O(\delta\varphi(q))$ characters $\chi \pmod q$ for which $|L(1/2,\chi\psi)M(\chi)| \geq \delta^{-1/2}(\log q)^{9/2}$. This follows if

$$\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} |L(1/2,\chi\psi)M(\chi)|^2 \ll \varphi(q)(\log q)^9, \tag{12}$$

and so it suffices to establish (12).

By the approximate functional equation we have for any primitive character χ (see e.g. [5, formula (2.2)])

$$L(1/2,\chi\psi) = \sum_{n=1}^{\infty}\frac{\chi(n)\psi(n)+\varepsilon(\chi\psi)\overline{\chi}(n)\overline{\psi}(n)}{\sqrt{n}}W\left(n\sqrt{\pi/(qD)}\right), \tag{13}$$

where W (denoted by V in [5]) is such that $W(y) = 1 + O(y^{10})$ and $W(y) \ll y^{-10}$. Using these bounds we see that

$$L(1/2,\chi\psi) = \sum_{n\leq(qD)^{\frac{3}{4}}}\frac{\chi(n)\psi(n)+\varepsilon(\chi\psi)\overline{\chi}(n)\overline{\psi}(n)}{\sqrt{n}}W\left(n\sqrt{\pi/(qD)}\right) + O\left(\frac{1}{qD}\right).$$

Using also the definition of $M(\chi)$ (see (5)) and the orthogonality of characters (adding back $\chi = \chi_0$), noting that $X(qD)^{3/4} \leq q$, we obtain

$$\begin{aligned}
&\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} |L(1/2,\chi\psi)M(\chi)|^2\\
&\ll \sum_{\chi \pmod q}|M(\chi)|^2\left|\sum_{n\leq(qD)^{\frac{3}{4}}}\frac{\chi(n)\psi(n)+\varepsilon(\chi\psi)\overline{\chi}(n)\overline{\psi}(n)}{\sqrt{n}}W\left(n\sqrt{\pi/(qD)}\right)\right|^2\\
&\quad+\frac{1}{q^2D^2}\sum_{\chi \pmod q}|M(\chi)|^2
\end{aligned}$$

$$\ll \varphi(q) \sum_{\substack{k_1,k_2 \le (qD)^{3/4} \\ \ell_1,\ell_2 \le X \\ k_1\ell_1 = k_2\ell_2}} \sum \frac{|(\mu * \mu\psi)(\ell_1)|\,|(\mu * \mu\psi)(\ell_2)|}{\sqrt{k_1\ell_1k_2\ell_2}} + \frac{\varphi(q)}{q^2D^2} \sum_{\substack{n \le X \\ D \nmid n}} \frac{|(\mu * \mu\psi)(n)|^2}{n}$$

$$\ll \varphi(q) \sum_{n \le X(qD)^{3/4}} \frac{d_3(n)^2}{n} + \frac{X^{\varepsilon}}{qD^2} \ll \varphi(q) \prod_{p \le X(qD)^{3/4}} \left(1 + \frac{3^2}{p}\right). \tag{14}$$

Now (12) follows from Mertens' theorem, and so the proof of Theorem 2 is completed.

The remaining task is to prove Proposition 4 which will be done in the following section.

4 Proof of Proposition 4

In this section we prove Proposition 4. By the Erdős-Turán inequality (see e.g. [10, Corollary 1.1] with $K = \lfloor q^{1/4} \rfloor$),

$$\left| \frac{1}{\varphi^+(q)} \sum_{\chi \pmod q}^{+} \mathbf{1}_{\theta_\chi \in (\alpha,\beta]} - (\beta - \alpha) \right| \le \frac{1}{q^{1/4}} + \frac{3}{\varphi^+(q)} \sum_{1 \le k \le q^{1/4}} \frac{1}{k} \left| \sum_{\chi \pmod q}^{+} e(k\theta_\chi) \right|.$$

Since $e(k\theta_\chi) = (\varepsilon(\chi)\varepsilon(\chi\psi))^k$, Proposition 4 follows immediately from the following lemma.

Lemma 5 *Let $k \in \mathbb{N}$. Let $D > 1$ be a square-free fundamental discriminant and let ψ be the associated primitive quadratic character modulo D. Let q be a prime with $q \nmid D$. Then*

$$\left| \sum_{\chi \pmod q}^{+} (\varepsilon(\chi)\varepsilon(\chi\psi))^k \right| \le \frac{1}{q^k} + 2k \cdot \frac{\varphi(q)}{q^{1/2}}.$$

Proof Our argument generalizes an argument in [2, Section 4] where the special case $k = 1$ was established (see [2, formula (4.6)]). Since $(q, D) = 1$, one gets (as pointed out in [2, page 603]) from (4) and the Chinese remainder theorem

$$\begin{aligned} \varepsilon(\chi\psi) &= \frac{1}{(Dq)^{1/2}} \sum_{\substack{a \pmod{Dq} \\ (a,Dq)=1}} \chi(a)\psi(a)e\left(\frac{a}{Dq}\right) \\ &= \frac{1}{(Dq)^{1/2}} \sum_{\substack{b \pmod D \\ (b,D)=1}} \sum_{\substack{c \pmod q \\ (c,q)=1}} \chi(bq + cD)\psi(bq + cD)e\left(\frac{bq + cD}{Dq}\right) \\ &= \chi(D)\psi(q)\varepsilon(\psi)\varepsilon(\chi). \end{aligned}$$

Hence

$$\varepsilon(\chi)\varepsilon(\chi\psi) = \chi(D)\psi(q)\varepsilon(\psi)\varepsilon(\chi)^2$$

and, using also (4),

$$\begin{aligned}\sideset{}{^+}\sum_{\chi \pmod q} (\varepsilon(\chi)\varepsilon(\chi\psi))^k \\ = \frac{\psi(q)^k\varepsilon(\psi)^k}{q^k} \sum_{\substack{a_1,\ldots,a_{2k} \pmod q \\ (a_j,q)=1}} e\left(\frac{a_1+\cdots+a_{2k}}{q}\right) \sideset{}{^+}\sum_{\chi \pmod q} \chi(D^k a_1\cdots a_{2k}).\end{aligned} \tag{15}$$

By orthogonality of characters we have, for any prime q and integers m, n such that $(mn, q) = 1$,

$$\begin{aligned}\sideset{}{^+}\sum_{\chi \pmod q} \chi(m)\overline{\chi}(n) &= \frac{1}{2} \sideset{}{^*}\sum_{\chi \pmod q} (1+\chi(-1))\chi(m)\overline{\chi}(n) \\ &= \frac{1}{2} \sum_{\chi \pmod q} (1+\chi(-1))\chi(m)\overline{\chi}(n) - 1 = \mathbf{1}_{q|(m\pm n)} \frac{\varphi(q)}{2} - 1.\end{aligned}$$

Applying this to (15), we obtain

$$\begin{aligned}\sideset{}{^+}\sum_{\chi \pmod q} (\varepsilon(\chi)\varepsilon(\chi\psi))^k = \frac{\psi(q)^k\varepsilon(\psi)^k}{2q^k}\varphi(q) \sum_{\substack{a_1,\ldots,a_{2k} \pmod q \\ D^k a_1\cdots a_{2k}\equiv\pm1 \pmod q}} e\left(\frac{a_1+\cdots+a_{2k}}{q}\right) \\ - \frac{\psi(q)^k\varepsilon(\psi)^k}{q^k} \sum_{\substack{a_1,\ldots,a_{2k} \pmod q \\ (a_j,q)=1}} e\left(\frac{a_1+\cdots+a_{2k}}{q}\right).\end{aligned} \tag{16}$$

The second term on the right hand side of (16) equals

$$-\frac{\psi(q)^k\varepsilon(\psi)^k}{q^k}\left(\sum_{a=1}^{q-1} e\left(\frac{a}{q}\right)\right)^{2k} = -\frac{\psi(q)^k\varepsilon(\psi)^k}{q^k},$$

and thus has absolute value at most $1/q^k$.

On the other hand, the first term on the right hand side of (16) equals

$$\frac{\psi(q)^k\varepsilon(\psi)^k}{2q^k}\varphi(q)\sum_{\ell=0}^{1} \sum_{\substack{a_1,\ldots,a_{2k-1} \pmod q \\ (a_j,q)=1}} e\left(\frac{a_1+\cdots+a_{2k-1}+(-1)^\ell\overline{D^k a_1\cdots a_{2k-1}}}{q}\right). \tag{17}$$

Here we have a $2k-1$-dimensional Kloosterman sum and by a bound of Smith [12, Theorem 6], the absolute value of (17) is

$$\leq \frac{\varphi(q)}{2q^k} \cdot 2 \cdot q^{(2k-1)/2} d_{2k}(q) = \frac{\varphi(q)}{q^{1/2}} d_{2k}(q).$$

Now the claim follows since q is a prime, so $d_{2k}(q) = 2k$. □

Acknowledgements The authors thank the anonymous referee for a careful reading of the manuscript and valuable comments. The first author was supported by Academy of Finland Grant no. 333707 and the second author was supported by Academy of Finland Grant no. 285894.

Funding Open Access funding provided by University of Turku (UTU) including Turku University Central Hospital.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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