

Intrinsic metrics in polygonal domains

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Abstract

We study inequalities between the hyperbolic metric and intrinsic metrics in convex polygonal domains in the complex plane. A special attention is paid to the triangular ratio metric in rectangles. A local study leads to investigation of the relationship between the conformal radius at an arbitrary point of a planar domain and the distance of the point to the boundary.

KEYWORDS

conformal radius, convex domain, distance to the boundary, hyperbolic metric, intrinsic metric, triangular ratio metric

MSC (2020)

51M09, 51M16, 30C20

1 | INTRODUCTION

During the past few decades, there has been considerable interest in the study of metrics defined in subdomains G of \mathbb{R}^n , $n \geq 2$. In the geometric function theory, the most useful metrics are *intrinsic metrics*. Distances in these metrics between two points measure not only how far the points are from each other, but also how close they are to the boundary of the domain. The hyperbolic distance $\rho_G(x, y)$, $x, y \in G$, of a planar domain G , is ideal for this purpose, in particular, because of its conformal invariance. There are many reasons to study metrics: Papadopoulos [14, pp. 42–48] lists 12 different metrics commonly used in the geometric function theory. We mention two factors motivating this research:

- (i) In the higher dimensional case $n \geq 3$ there is no metric, equally flexible as the hyperbolic metric is in the planar case. Thus, many authors have introduced metrics of the hyperbolic type which share some but not all properties of the hyperbolic metric [11].
- (ii) In the planar case, estimating the hyperbolic distances of a given planar domain is a difficult problem [7]. Therefore, finding concrete estimates for the hyperbolic metric in terms of simpler metric is needed; these estimates should take into account both the geometric structure and the metric properties of the boundary.

We study here the topic (ii), continuing the earlier work [8, 11, 16, 17]. In particular, we investigate the case when the domain is a polygonal domain in the plane and compare the value of the hyperbolic metric to the triangular ratio metric s_G [11], see Section 2 for the definition. In the case, when the polygonal domain G is a rectangle, our main result is Theorem 4.3. We also formulate a conjecture about the sharp constant in this theorem (Conjecture 4.4).

It also turns out that the methodology of our proofs leads to the study of the *conformal radius* of a domain and its connection with the distance to the boundary. In Section 5, we study some concrete domains and find the maximum

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of their conformal radii. In Section 6, we investigate the ratio $2d_G(u)/r_G(u)$ for $u \in G$, where G is a polygonal domain, $d_G(u) = \text{dist}(u, \partial G)$, ∂G denotes the boundary of G , and $r_G(u)$ is the conformal radius of G at u . Also, we prove that the maximum of the ratio is attained on some graph which consists of those points $u \in G$ for which $d_G(u) = |u - z_j|$, $j = 1, 2$ for at least two distinct points $z_1 \neq z_2$, $z_j \in \partial G$ (Theorem 6.1). Then, we give some examples of polygonal domains and calculate the maximal value of $2d_G(u)/r_G(u)$; the obtained values give the lower estimates for the best constant $C = C(G)$ in the inequality

$$\text{th} \frac{\rho_G(u, v)}{2} \leq Cs_G(u, v), \quad u, v \in G.$$

2 | PRELIMINARIES

Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plane. Also, let \mathbb{B}^2 be the unit disk and \mathbb{H}^2 be the upper half-plane in the complex plane \mathbb{C} .

2.1 | Möbius transformations

A Möbius transformation is a mapping of the form

$$z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d, z \in \mathbb{C}, \quad ad - bc \neq 0.$$

The special Möbius transformation [2]

$$T_a(z) = \frac{z - a}{1 - \bar{a}z}, \quad a \in \mathbb{B}^2 \setminus \{0\}, \quad (1)$$

maps the unit disk \mathbb{B}^2 onto itself with $T_a(a) = 0$, $T_a(\pm a/|a|) = \pm a/|a|$, where \bar{a} is the complex conjugate of a .

Suppose that G is a proper subdomain in \mathbb{R}^2 .

Denote by $d_G(x)$ the Euclidean distance $\text{dist}(x, \partial G) = \inf\{|x - z| : z \in \partial G\}$ between the point x and the boundary of G . The triangular ratio metric $s_G : G \times G \rightarrow [0, 1]$ is defined by [11, 17]

$$s_G(x, y) = \frac{|x - y|}{\inf_{z \in \partial G} (|x - z| + |z - y|)}, \quad x, y \in G. \quad (2)$$

Using the above notation, we can also record the formulas for the hyperbolic metric ρ_G in \mathbb{H}^2 , and \mathbb{B}^2 as [6, 7, 11, Equation (4.8), p. 52 and Equation (4.14), p. 55]

$$\text{ch} \rho_{\mathbb{H}^2}(x, y) = 1 + \frac{|x - y|^2}{2d_{\mathbb{H}^2}(x)d_{\mathbb{H}^2}(y)}, \quad x, y \in \mathbb{H}^2,$$

$$\text{sh}^2 \frac{\rho_{\mathbb{B}^2}(x, y)}{2} = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}, \quad x, y \in \mathbb{B}^2.$$

Note that the hyperbolic metric enjoys the following *conformal invariance* property: if G is a domain and $f : G \rightarrow G' = f(G)$ is a conformal mapping, then

$$\rho_G(x, y) = \rho_{G'}(f(x), f(y)).$$

Thus, the hyperbolic metric ρ_G can be defined in any planar simply connected domain $G \neq \mathbb{C}$ in terms of a conformal mapping of the domain onto the unit disk [7, Thm. 6.3, p. 26].

In terms of complex numbers, the formulas of the hyperbolic metric for the two cases \mathbb{H}^2 and \mathbb{B}^2 can be simplified to

$$\operatorname{th} \frac{\rho_{\mathbb{H}^2}(x, y)}{2} = \operatorname{th} \left(\frac{1}{2} \log \left(\frac{|x - \bar{y}| + |x - y|}{|x - \bar{y}| - |x - y|} \right) \right) = \left| \frac{x - y}{x - \bar{y}} \right|,$$

$$\operatorname{th} \frac{\rho_{\mathbb{B}^2}(x, y)}{2} = \operatorname{th} \left(\frac{1}{2} \log \left(\frac{|1 - x\bar{y}| + |x - y|}{|1 - x\bar{y}| - |x - y|} \right) \right) = \left| \frac{x - y}{1 - x\bar{y}} \right|.$$

2.2 | Elliptic integrals and Jacobi elliptic functions

Let $K(\lambda)$, $0 < \lambda < 1$, denote the complete elliptic integral of the first kind [3],

$$K(\lambda) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\lambda^2 t^2)}}, \quad K'(\lambda) = K(\sqrt{1-\lambda^2}). \quad (3)$$

Let $\operatorname{sn}(\cdot, \lambda)$, $\operatorname{cn}(\cdot, \lambda)$, and $\operatorname{dn}(\cdot, \lambda)$ denote the Jacobi elliptic functions. We recall that, by definition, $\operatorname{sn}(z, \lambda) = \sin \operatorname{am}(z, \lambda)$, $\operatorname{cn}(z, \lambda) = \cos \operatorname{am}(z, \lambda)$, and $\operatorname{dn}(z, \lambda) = \frac{d}{dz} \operatorname{am}(z, \lambda)$, where $\operatorname{am}(z, \lambda)$ is the Jacobi amplitude, that is, the function $\varphi = \operatorname{am}(z, \lambda)$ inverse to the incomplete elliptic integral of the first kind [1]

$$z = \int_0^\varphi \frac{d\theta}{\sqrt{1 - \lambda^2 \sin^2 \theta}}.$$

Elliptic integrals and elliptic functions occur in conformal mapping of a rectangle onto the upper half-plane [1, p. 358].

2.3 | Conformal radius

The *conformal radius* $r_D(z_0)$ of a simply connected domain D with a non-degenerate boundary at a point $z_0 \neq \infty$ is a well-known characteristic playing an important role in geometric function theory and applications. By definition, $r_D(z_0)$ is the radius of a disk centered at the origin which can be mapped onto D conformally by a function F with the normalization $F(0) = z_0$ and $F'(0) = 1$ (see, e.g., [4, 5, 10, 15, 18]). Let $g : D \rightarrow \mathbb{B}^2$ be a conformal mapping of D onto \mathbb{B}^2 and f be its inverse. If $g(z_0) = \zeta$, then [15, p. 162]

$$r_D(z_0) = |f'(\zeta)|(1 - |\zeta|^2) = \frac{1 - |g(z_0)|^2}{|g'(z_0)|}, \quad (4)$$

and if $G : D \rightarrow \mathbb{H}^2$ is a conformal mapping of D onto the upper half-plane, F is its inverse and $G(z_0) = w$, then

$$r_D(z_0) = 2 \operatorname{Im} w |F'(w)| = \frac{2 \operatorname{Im} G(z_0)}{|G'(z_0)|}. \quad (5)$$

It is well known that the conformal radius $r_D(z_0)$ and the distance to the boundary $d_D(z_0)$ are equivalent in the sense that for every simply connected domain $D \subsetneq \mathbb{C}$

$$d_D(z_0) \leq r_D(z_0) \leq 4d_D(z_0). \quad (6)$$

The left-hand side of Equation (6) follows from the Schwarz lemma, and the right-hand one is a corollary of the Koebe quarter theorem. It should be noted that both inequalities (6) above are sharp. The left-hand side is sharp when $f(\zeta) = \zeta$ and $D = \mathbb{B}^2$. Also, the Koebe function $k(\zeta) := \zeta/(1 - \zeta)^2$ exhibits equality in the right-hand side. Since $k(\zeta)$ maps the open unit disk \mathbb{B}^2 onto $D := \mathbb{C} \setminus (-\infty, -1/4]$, we get $r_D(0) = 1 = 4d_D(0)$.

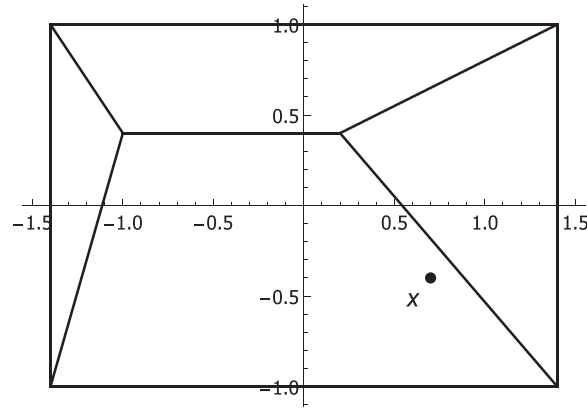


FIGURE 1 For a fixed point x in a rectangle, there are four subdomains with the following property. If y is a point of a subdomain, then the point z providing the minimum of $|x - z| + |z - y|$ is on the common side of the subdomain and of the rectangle.

Moreover, for a convex domain D and an arbitrary point $z_0 \in D$

$$d_D(z_0) \leq r_D(z_0) \leq 2d_D(z_0), \quad (7)$$

where both the inequalities are sharp. In the first inequality, by the Schwarz lemma, the equality holds if and only if D is a disk and z_0 is its center; in the second inequality, we have equality if and only if D is a half-plane and z_0 is its arbitrary point (see, e.g., [9, Chap. 2, Section 2.5, Theorem 2.15], [12, Chap.1, Section 1.4, Theorem 1.8]).

3 | NEAREST POINT FOR A RECTANGLE

Consider the rectangle R with vertices $\pm k \pm i$, where $k \geq 1$. From the definition (2) of the s -metric, we see that if a point $x \in R$ is fixed, then the infimum $\inf_{z \in \partial G} (|x - z| + |z - y|)$ is attained at some points z lying on a side of ∂R . This side depends on the location of y .

In Figure 1, we show, for a fixed $x \in R$, the subdomains of R such that if y belongs to one of these subdomains, then z belongs to the corresponding side of R which is a common side of R and of the subdomain. Among the four subdomains, two are trapezoids, and the other two are triangles. The lower and the upper subdomains are trapezoids if and only if $\operatorname{Re} x^2 - \operatorname{Im} x^2 < k^2 - 1$; in the case, $\operatorname{Re} x^2 - \operatorname{Im} x^2 > k^2 - 1$ trapezoids are the left and right subdomains. The case $\operatorname{Re} x^2 - \operatorname{Im} x^2 = k^2 - 1$ corresponds to a subdivision of R into four triangles. In Figure 1, we show a subdivision for $k = 1.4$, $x = 0.7 - 0.4i$.

We can also describe the geometry of disks and circles in the s -metric. Every circle centered at x of radius r is either a Euclidean circle or a piecewise smooth Jordan curve. In Figure 2, we show such curves for the case $k = 1.4$, $x = 0.7 - 0.4i$, $r = 0.1j$, $1 \leq j \leq 9$. We note that the corner points of these curves are on the segments separating R into four parts described above (see Figure 1). Besides, the disks are convex sets in the Euclidean metric.

4 | HYPERBOLIC METRIC VERSUS TRIANGLE RATIO METRIC IN RECTANGLE

Our goal is to find for the rectangle R the best constants $C_1, C_2 > 0$ such that the inequality

$$C_1 s_R(u, v) \leq \operatorname{th} \frac{\rho_R(u, v)}{2} \leq C_2 s_R(u, v), \quad u, v \in R, \quad (8)$$

holds. Now, we will find the limit

$$\lim_{u \rightarrow v} \frac{\operatorname{th} \frac{\rho_R(u, v)}{2}}{s_R(u, v)},$$

and give its geometric interpretation.

FIGURE 2 Some s -metric circles in the rectangle $R = [-1.4, 1.4] \times [-1, 1]$ centered at $x = 0.7 - 0.4i$ with radii $r = 0.1j$, $1 \leq j \leq 9$.

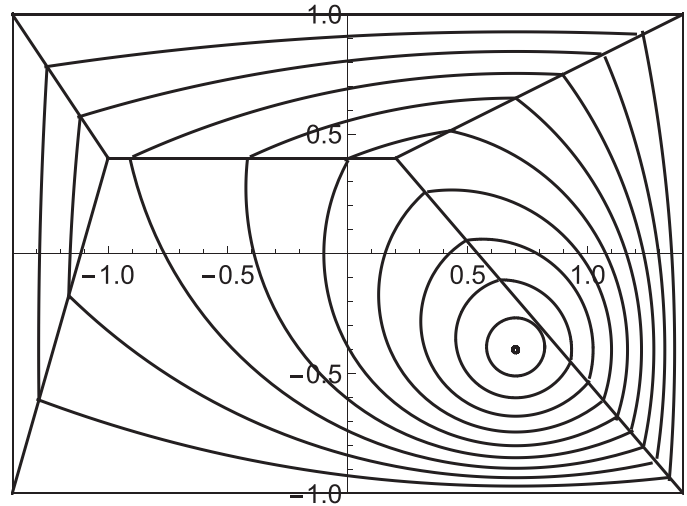
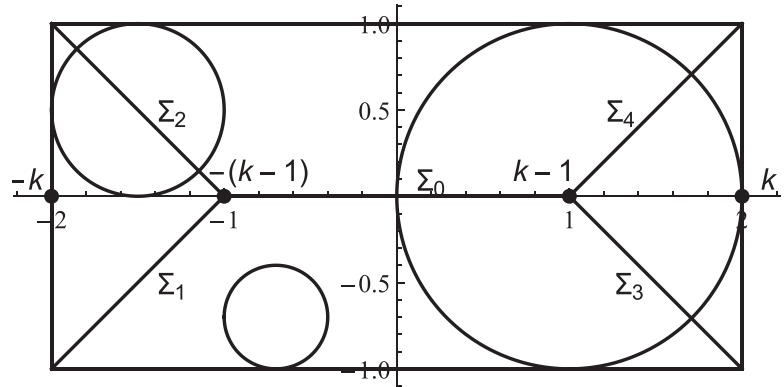


FIGURE 3 Decomposition of a rectangle R and the five segments Σ_j , $0 \leq j \leq 4$, $k = 2$.



Consider the segments Σ_1 and Σ_2 which are parts of the bisectors of the angles at the vertices $-k - i$ and $-k + i$ of R , respectively, connecting these vertices with the point $-(k - 1)$ on the real axis; similarly, let also Σ_3 and Σ_4 join the vertices $k - i$ and $k + i$ with the point $k - 1$ on the real axis. Consider also the segment Σ_0 with endpoints $\pm(k - 1)$. These five segments subdivide R into four subdomains, two trapezoids and two triangles (see Figure 3); in the case $k = 1$, the trapezoids degenerate into triangles and Σ_0 degenerates into a point.

We see that if a point v is in one of these subdomains, then the point ζ_0 , where the minimal value of $|\zeta - v|$, $\zeta \in \partial R$, is attained, lies on the side of R which is common with this subdomain. If v is on one of the segments described above and $v \neq \pm(k - 1)$, then the shortest distance is attained at two sides of R . At last, if $v = \pm(k - 1)$, then there are three sides where the minimal value of $|\zeta - v|$, $\zeta \in \partial R$, is attained; in that case $k = 1$, we have the square, the points $\pm(k - 1)$ coincide with its center and the minimal distance is attained at all four of the sides.

We can describe this situation as follows. If v is an interior point of one subregion, then the largest disk contained in R and centered at v touches ∂R at one point lying on the boundary of R which is common with this subdomain. If $v \in \cup_{j=0}^4 \Sigma_j$ and $v \neq \pm(k - 1)$, then the largest disk touches two of the sides, and if $v = \pm(k - 1)$, then it touches ∂R at three points. These three cases are shown in Figure 3. (If $k = 1$, then the largest disk touches all the four sides of R .)

Lemma 4.1. *We have*

$$\lim_{u \rightarrow v} \frac{\operatorname{th} \frac{\rho_R(u, v)}{2}}{s_R(u, v)} = \frac{2d_R(v)}{r_R(v)}, \quad (9)$$

where $d_R(v) = \operatorname{dist}(v, \partial R)$, and $r_R(v)$ is the conformal radius of R at the point v .

Proof. Let $f : R \rightarrow \mathbb{H}^2$ be a conformal mapping of R onto the upper half-plane \mathbb{H}^2 . Then, by the definition,

$$\operatorname{th} \frac{\rho_R(u, v)}{2} = \frac{|f(u) - f(v)|}{|f(u) - \overline{f(v)}|}.$$

Let v be a point from the lower trapezoidal part of R (other three cases can be considered similarly). Then,

$$s_R(u, v) = \frac{|f(u) - \overline{f(v)}|}{|u - \bar{v} + 2i|}.$$

Therefore,

$$\frac{\operatorname{th} \frac{\rho_R(u, v)}{2}}{s_R(u, v)} = \frac{|f(u) - f(v)|}{|u - v|} \frac{|u - \bar{v} + 2i|}{|f(u) - \overline{f(v)}|} \rightarrow 2(\operatorname{Im}(v) + 1) \frac{|f'(v)|}{2 \operatorname{Im} f(v)}.$$

Since

$$\operatorname{Im}(v) + 1 = d_R(v) \quad \text{and} \quad \frac{|f'(v)|}{2 \operatorname{Im} f(v)} = \frac{1}{r_R(v)},$$

we obtain Equation (9). □

We see that if Equation (8) holds, then by Lemma 4.1 we get

$$\frac{2}{C_2} d_R(v) \leq r_R(v) \leq \frac{2}{C_1} d_R(v).$$

Therefore, we obtain an estimation of the conformal radius $r_R(v)$ through $d_R(v)$. On the other hand, if for some point $v \in R$ we have,

$$r_R(v) = \tilde{C} d_R(v), \quad 0 < \tilde{C} \leq 1,$$

then the constant C_2 in Equation (8) satisfies $C_2 \geq 2/\tilde{C}$. From Equation (6), we obtain

$$\frac{1}{2} \leq \frac{2d_D(v)}{r_D(v)} \leq 2, \quad v \in D,$$

and for convex domains, with the help of Equation (7), we deduce that

$$1 \leq \frac{2d_D(v)}{r_D(v)} \leq 2, \quad v \in D.$$

Given $k \geq 1$, let $\lambda \in (0, 1)$ be the unique root of the equation

$$\frac{K(\lambda)}{K'(\lambda)} = \frac{k}{2}, \tag{10}$$

where $K(\lambda)$ is as defined in Equation (3).

Lemma 4.2. *Let D be a convex planar domain with a nonempty boundary ∂D and suppose that one of the following conditions is valid:*

(i) *The circle centered at a point $z_0 \in D$ of radius $d_D(z_0)$ touches ∂D at some point ζ_0 ;*

(ii) The boundary ∂D contains two linear segments $[a, \zeta_0]$ and $[\zeta_0, b]$ and the circle centered at a point $z_0 \in D$ of radius $d_D(z_0)$ touches ∂D at two points $\zeta_1 \in [a, \zeta_0]$ and $\zeta_2 \in [\zeta_0, b]$.

Then for every z_1 which is an interior point of the segment with endpoints z_0, ζ_0 we have

$$\frac{d_D(z_1)}{r_D(z_1)} \leq \frac{d_D(z_0)}{r_D(z_0)}. \quad (11)$$

Moreover, if either (i) holds and D is distinct from a half-plane, or (ii) is valid and ∂D is distinct from the union of two rays coming from the point ζ_0 and passing through the points a and b , then the inequality in Equation (11) is strict.

Proof. Since d_D and r_D are invariant under the shifts of the plane, without loss of generality we can assume that $\zeta_0 = 0$.

Let $\alpha = z_0/z_1, \alpha > 1$. Denote $G = \psi(D)$, where the mapping $\psi : z \mapsto \alpha z$. We note that $\psi(z_1) = z_0$ and $r_G(z_0) = \alpha r_D(z_1)$, $d_G(z_0) = \alpha d_D(z_1)$, therefore, $d_D(z_1)/r_D(z_1) = d_G(z_0)/r_G(z_0)$. But because D is convex, we obtain $D \subset G$, therefore, $r_G(z_0) \geq r_D(z_0)$. Since, evidently $d_G(z_0) = d_D(z_0)$, we obtain Equation (11). If either (i) holds and D is distinct from a half-plane, or (ii) is valid and ∂D is distinct from the union of two rays coming from the point ζ_0 and passing through the points a and b , then $D \neq G$ and, therefore, $r_G(z_0) < r_D(z_0)$, so the inequality in Equation (11) is strict. \square

Theorem 4.3. If R is the rectangle $[-k, k] \times [-1, 1]$, $k \geq 1$, then for all $v \in R$

$$1 < \frac{2d_R(v)}{r_R(v)} \leq C(\lambda), \quad (12)$$

where

$$C(\lambda) = K(\lambda) \left| \frac{\operatorname{cn}(iK(\lambda), \lambda) \operatorname{dn}(iK(\lambda), \lambda)}{\operatorname{sn}(iK(\lambda), \lambda)} \right|. \quad (13)$$

Both inequalities of Equation (12) are sharp. Moreover, equality in the second inequality holds if and only if $v = \pm(k-1)$.

Proof. The lower estimation in Equation (12) is evident. We will prove the upper one. First, we will show that the maximal value of $2d_R(v)/r_R(v)$, $v \in R$, is attained only on the segment with endpoints $\pm(v-1)$. Using the statement (i) of Lemma 4.2, we see that the maximum of the value $2d_R(v)/r_R(v)$, $v \in R$, is attained if $v \in \cup_{j=0}^4 \Sigma_j$. By the statement (ii) of Lemma 4.2 we conclude that v is on the segment Σ_0 with endpoints $\pm(k-1)$.

For $v \in \Sigma_0$ we have $d_R(v) = 1$. Therefore, to find the maximal value of $2d_R(v)/r_R(v)$ we only need to find the minimal value of the conformal radius $r_R(v)$, $v \in \Sigma_0$. But $r_R(v)$ is an even function on $[-k, k]$ which is strictly decreasing on $[0, k]$. Therefore, the minimal value of $r_R(v)$ on $\Sigma_0 = [-(k-1), k-1]$ is attained at the endpoints of the segment, that is, at the points $\pm(k-1)$.

Now, we will find the value $r_R(k-1)$. To calculate it, we replace, for convenience, R with its image $R_2 = [-\alpha, \alpha] \times [0, 2\alpha k]$ under the mapping $z \mapsto i\alpha(k-z)$, where the constant $\alpha > 0$ will be fixed below; we note that, under the mapping, the point $k-1$ goes to $i\alpha$. This mapping does not change the value of the conformal radius. It is known that R_2 can be mapped onto the upper half-plane \mathbb{H}^2 such that its vertices $\alpha(-1+2ik)$, $-\alpha$, α , and $\alpha(1+2ik)$ will go to $-1/\lambda$, -1 , 1 , and $1/\lambda$, respectively. Here, $\lambda \in (0, 1)$ is the unique root of Equation (10). If we put $\alpha = K(\lambda)$, then we can use as such a mapping the Jacobi elliptic function, $f(z) = \operatorname{sn}(z, \lambda)$ which maps R_2 onto \mathbb{H}^2 with the correspondence of boundary points indicated above [1, p. 358]. By the definition of the conformal radius, we have

$$r_{R_2}(iK(\lambda)) = \frac{2 \operatorname{Im} f(iK(\lambda))}{|f'(iK(\lambda))|}.$$

Since $f'(z) = \operatorname{cn}(z, \lambda) \operatorname{dn}(z, \lambda)$ and $f(iK(\lambda))$ is a pure imaginary number, we find

$$r_R(k-1) = r_{R_2}(iK(\lambda)) = \left| \frac{2 \operatorname{sn}(iK(\lambda), \lambda)}{\operatorname{cn}(iK(\lambda), \lambda) \operatorname{dn}(iK(\lambda), \lambda)} \right|.$$

We also have $d_{R_2}(iK(\lambda)) = K(\lambda)$. Since the ratio of the distance to the boundary and the conformal radius is invariant under linear conformal mappings, we have,

$$\frac{2d_R(k-1)}{r_R(k-1)} = K(\lambda) \left| \frac{\operatorname{cn}(iK(\lambda), \lambda) \operatorname{dn}(iK(\lambda), \lambda)}{\operatorname{sn}(iK(\lambda), \lambda)} \right|.$$

As we showed above, this is the biggest value of $2d_R(v)/r_R(v)$ over R . □

Since $\operatorname{sn}(iK(\lambda), \lambda)$ is a pure imaginary number, $\operatorname{cn}(z, \lambda) = \sqrt{1 - \operatorname{sn}^2(z, \lambda)}$ and $\operatorname{dn}(z, \lambda) = \sqrt{1 - \lambda^2 \operatorname{sn}^2(z, \lambda)}$, we can write Equation (13) in the form

$$C(\lambda) = K(\lambda) \frac{\sqrt{(1 + a^2(\lambda))(1 + \lambda^2 a^2(\lambda))}}{a(\lambda)}, \quad \text{where } a(\lambda) := |\operatorname{sn}(iK(\lambda), \lambda)|.$$

Corollary 4.4. *We have*

$$\frac{2}{C(\lambda)} d_R(v) \leq r_R(v) < 2d_R(v), \quad v \in R,$$

and the inequalities are sharp.

Conjecture 4.5. *Let $C(\lambda)$ be defined as in Equation (13). Then for all points u, v lying in the rectangle $R = [-k, k] \times [-1, 1]$, $k \geq 1$, we have*

$$s_R(u, v) \leq \operatorname{th} \frac{\rho_R(u, v)}{2} \leq C(\lambda) s_R(u, v). \quad (14)$$

These inequalities are sharp.

Remark 4.6. The inequality $s_R(u, v) \leq \operatorname{th}(\rho_R(u, v)/2)$ and its sharpness are evident.

Above, we considered rectangles $R = [-k, k] \times [-1, 1]$ with $k \geq 1$. This corresponds to the values $\lambda \geq \lambda_0 := 3 - 2\sqrt{2} = 0.171572875 \dots$, where λ_0 is the unique root of Equation (10) for $k = 1$ [3, p. 81, Table 5.1]. We can also consider a rectangle with $k < 1$. It is evident that this case is easily reduced to the case $k \geq 1$ because we can apply the mapping $z \mapsto (i/k)z$ which maps R onto the rectangle $[-1/k, 1/k] \times [-1, 1]$. For such a rectangle, we replace λ with $h(\lambda) = (1 - \sqrt{\lambda})^2 / (1 + \sqrt{\lambda})^2$. Therefore, let us define $\tilde{C}(\lambda)$ as follows:

$$\tilde{C}(\lambda) := \begin{cases} C(h(\lambda)), & 0 < \lambda < \lambda_0; \\ C(\lambda), & \lambda_0 \leq \lambda < 1. \end{cases}$$

Then

$$r_R(v) \leq 2d_R(v) \leq \tilde{C}(\lambda) s_R(u, v).$$

The graph of $\tilde{C}(\lambda)$ for $0 < \lambda < 1$ is shown in Figure 4.

We have

$$\tilde{C}(\lambda_0) = \varphi(\lambda_0) := \frac{(1 + \lambda_0) K'(\lambda_0)}{2} = 1.854074677 \dots, \quad (15)$$

and

$$\lim_{\lambda \rightarrow 0^+} \tilde{C}(\lambda) = \lim_{\lambda \rightarrow 1^-} \tilde{C}(\lambda) = \frac{\pi}{2} \coth \frac{\pi}{2} = 1.7126885749 \dots$$

The first limiting case corresponds to a square, and the second one matches to the case of a half-strip.

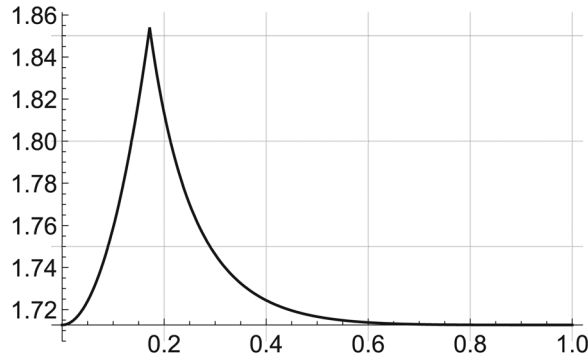


FIGURE 4 The graph of the function $\tilde{C}(\lambda)$.

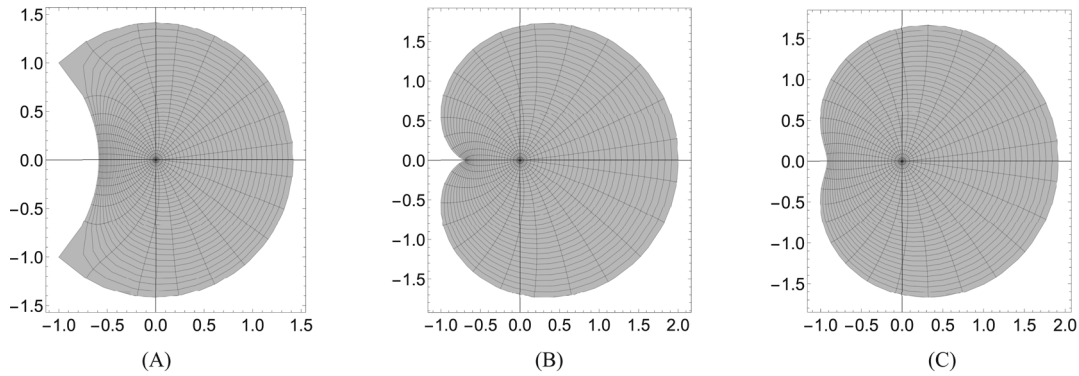


FIGURE 5 (a) The graph of $\phi_2(\mathbb{B}^2)$. (b) The graph of $\phi_3(\mathbb{B}^2)$ for $\alpha = 4/3$ and $\beta = 2/3$. (c) The graph of $\phi_4(\mathbb{B}^2)$.

5 | CONFORMAL RADIUS OF SPECIFIC DOMAINS AND ITS MAXIMUM

In this section, we determine the maximal value of the conformal radius for some domains which are conformal images of the unit disk \mathbb{B}^2 . These domains can be found in [13], and references therein. We note that exact values of the maximal value for conformal radius of some special domains can be found in [15, tables of some functionals].

Example 5.1. Consider the function $w = \phi_1(z) = \sqrt{1+z} - 1, z \in \mathbb{B}^2$, where the branch of the square root is chosen such that $\sqrt{1} = 1$. The function ϕ_1 maps the open unit disk \mathbb{B}^2 onto the domain, D_1 which is the right-half of the lemniscate of Bernoulli. Hence, the inverse mapping $z = \varphi_1(w) = w(w+2)$ maps D_1 onto \mathbb{B}^2 . Since $\varphi_1(0) = 0$ and $\varphi_1'(0) = 2 > 0$, we have $r_{\phi_1(\mathbb{B}^2)}(0) = 0.5$.

Now, we will find the maximum of the conformal radius of D_1 . If $w = \phi_1(z)$, where $z = te^{i\theta}, t \in [0, 1)$ and $\theta \in [0, 2\pi)$, then by (4) the conformal radius of D_1 at the point w is equal to

$$r_{D_1}(w) = \frac{1}{2}|1+z|^{-1/2}(1-|z|^2) \leq \frac{1}{2}(1-t)^{-1/2}(1-t^2) = \frac{1}{2}(1-t)^{1/2}(1+t),$$

and the equality holds if and only if $\theta = \pi$. The maximal value of the function $(1-t)^{1/2}(1+t)/2$ is attained at $t_0 = 1/3$ and is equal to $2\sqrt{6}/9$. Thus, we see that the maximum of $r_{D_1}(w)$ in \mathbb{B}^2 is equal to $2\sqrt{6}/9 = 0.544331 \dots$ which is attained at the point $w = \phi_1(-1/3) = \sqrt{6}/3 - 1 = -0.183503 \dots$

Example 5.2. Consider the function $w = \phi_2(z) = z + \sqrt{1+z^2} - 1, z \in \mathbb{B}^2$, where the branch of the square root is chosen such that $\sqrt{1} = 1$. We note that this function is closely connected with the function inverse to the Joukowski transform. It maps the unit disk \mathbb{B}^2 onto the crescent-shaped region D_2 , see Figure 5a. The inverse function $z = \varphi_2(w) = (w+1 - (1/(1+w)))/2$ maps D_2 onto \mathbb{B}^2 . Hence, $\varphi_2(0) = 0, \varphi_2'(0) = 1$, therefore $r_{D_2}(0) = 1$.

Now, we will find the maximal value of the conformal radius $r_{D_2}(w)$ for points w lying on the real axis. Let $w = \phi_2(t)$, $t \in (-1, 1)$, then, with the help of Equation (4), we obtain

$$r_{D_2}(w) = \left| 1 + \frac{t}{\sqrt{1+t^2}} \right| (1 - |z|^2) \leq \left(1 + \frac{|t|}{\sqrt{1+t^2}} \right) (1 - t^2).$$

Simple analysis shows that the maximal value of the expression on the right-hand side equals $M = 1.17117 \dots$, it is attained at the point $t_0 = 0.319968 \dots$ which is the minimal positive root of the equation $4t^8 + 12t^6 + t^4 - 10t^2 + 1 = 0$. The corresponding point of maximum of the conformal radius is equal to $w = \phi_2(t_0) = 0.369911 \dots$. Therefore, we found that $\max_{-1 < w < 1} r_{D_2}(w) = M$. Finding the maximum of $r_{D_2}(w)$ over all $w \in D_2$ is an open problem.

Example 5.3. Consider the analytic function $w = \phi_3(z) = \alpha z + \beta z^2$, $z \in \mathbb{B}^2$, $\alpha, \beta \in \mathbb{C}$ and $|\alpha| \geq 2|\beta| > 0$. This function maps the unit disk \mathbb{B}^2 onto the domain D_3 bounded by a heart-shaped curve, see Figure 5b for $\alpha = 4/3$ and $\beta = 2/3$. The maximal value of the conformal radius $r_{D_3}(w)$ equals $\ell(r_1)$, where

$$\ell(r) := |\alpha| + 2|\beta|r - |\alpha|r^2 - 2|\beta|r^3,$$

and

$$r_1 = \frac{-|\alpha| + \sqrt{|\alpha|^2 + 12|\beta|^2}}{6|\beta|};$$

the maximal value is attained at the point $w_0 = \phi_3(r_1 e^{i\delta})$ where $\delta = \arg(\alpha/\beta)$. For example, if we let $\alpha = \sqrt{2}$ and $\beta = 1/2$, then the function $w = \phi_4(z) = \sqrt{2}z + z^2/2$, $z \in \mathbb{B}^2$ maps the unit disk \mathbb{B}^2 onto the domain D_4 bounded by a limaçon, see Figure 5c. The maximal value of the conformal radius $r_{D_4}(w)$ equals $(2/27)(7\sqrt{2} + 5\sqrt{5}) = 1.56147 \dots$, it is attained at the point $w_0 = \phi_4((\sqrt{5} - \sqrt{2})/3) = (4\sqrt{10} - 5)/18 = 0.424951 \dots$

Example 5.4. Consider the univalent function $w = \phi_5(z) = z/(1 - \alpha z^2)$, $z \in \mathbb{B}^2$, $0 \leq \alpha < 1$. This function ϕ_5 maps the unit disk \mathbb{B}^2 onto the domain D_5 , bounded by a Booth lemniscate. We have

$$\phi_5'(z) = \frac{1 + \alpha z^2}{(1 - \alpha z^2)^2},$$

therefore, for $w = \phi_5(z)$ we have

$$r_{D_5}(w) = \left| \frac{1 + \alpha z^2}{(1 - \alpha z^2)^2} \right| (1 - |z|^2) \leq \psi(t^2) := \frac{1 + \alpha t^2}{(1 - \alpha t^2)^2} (1 - t^2), \quad t = |z|.$$

Now, we will investigate the behavior of $\psi(\tau)$, $\tau \in [0, 1]$. Simple analysis shows that for $0 \leq \alpha \leq 1/3$ the function ψ decreases on $[0, 1]$. Therefore, $\psi(t^2) \leq \psi(0) = 1$ and the maximal value of the conformal radius $r_{D_5}(w)$ equals 1; it is attained at the origin.

If $1/3 < \alpha < 1$, then ψ has a unique point of maximum $\tau_\alpha = (3\alpha - 1)/(\alpha(3 - \alpha))$. Thus, in this case, the maximal value $m(\alpha)$ of the conformal radius $r_{D_5}(w)$ equals $\psi(\tau_\alpha)$, that is,

$$m(\alpha) = \frac{(1 + \alpha)^2}{8\alpha(1 - \alpha)}.$$

It is attained at the points

$$w_0 = \pm \phi_5(\tau_\alpha^2) = \pm \frac{(3\alpha - 1)(3 - \alpha)}{(1 - \alpha)(-\alpha^2 + 14\alpha - 1)}.$$

6 | COMPARISON OF THE CONFORMAL RADIUS AND THE DISTANCE TO THE BOUNDARY FOR SOME DOMAINS

It is easy to see that we can apply the above method, based on Lemma 4.2, to obtain sharp two-sided estimations of $2d_D(\cdot)/r_D(\cdot)$ for other convex planar domains D .

First, consider a bounded convex n -gon P and its decomposition into parts: $P = \cup_{j=1}^n P_j$ such that $z \in P_j$ if and only if z is closer to the j th side of P , than to others. Then, we investigate the set $\cup_{j=1}^n \partial P_j$. It is a graph and after removing from it the boundary points of P and open, that is, without endpoints, edges which have nonempty intersection with ∂P we obtain its subgraph; denote it by $Gr(P)$. For example, if we take as P a rectangle $R = [-k, k] \times [-1, 1]$, then $Gr(P)$ consists of the segment Σ_0 (see Section 4). If P admits an inscribed circle (a circle which is tangent to all sides of P), then $Gr(P)$ consists of the point which is the center of the circle.

Lemma 4.2 immediately implies the following result.

Theorem 6.1. *The maximum of $2d_P(u)/r_P(u)$, $u \in P$, is attained at some point of the graph $Gr(P)$.*

Analyzing some concrete domains and classes of domains (see the examples below), we can suggest the following conjecture.

Conjecture 6.2. *The maximum of $2d_P(u)/r_P(u)$, $u \in P$, is attained at a vertex of the graph $Gr(P)$.*

Remark 6.3. We note that the statement of Theorem 6.1 is also valid for most unbounded convex polygonal domains P , excluding those with degenerate (empty) $Gr(P)$ such as strips and infinite sectors. Actually, if P is unbounded, then ∂P contains two rays, L_1 and L_2 . If the rays are not parallel, for simplicity, we can assume that their continuations intersect at the origin. Consider the set \mathcal{A} of points in P such that maximal disks centered at these points and contained in P touch both the rays; it is also a ray coming from a point z_0 . Then applying the mappings $z \mapsto \alpha z$, $\alpha > 0$, we show as in the proof of Lemma 4.2 that when approaching the origin by the set \mathcal{A} the value of $2d_P(u)/r_P(u)$ increases. Therefore, the maximal value of $2d_P(u)/r_P(u)$, $u \in \mathcal{A}$, is attained at the vertex z_0 of $Gr(P)$. If L_1 and L_2 are parallel, then the ray \mathcal{A} is parallel to them, and we can apply shifts $z \mapsto z + h$ instead of the mappings $z \mapsto \alpha z$, $\alpha > 0$.

Example 6.4. Convex sector $S_\gamma := \{z \in \mathbb{C} : 0 < \arg z < \gamma\}$, $0 < \gamma < \pi$; The upper estimate for the value of $2d_{S_\gamma}(\cdot)/r_{S_\gamma}(\cdot)$ is attained at every point of its bisector. The function $g(z) = (z^{\pi/\gamma} - i)/(z^{\pi/\gamma} + i)$ maps the sector S_γ onto the unit disk \mathbb{B}^2 so that the point $e^{i\gamma/2}$ goes to the origin. Using formula (4), we obtain

$$r_{S_\gamma}(e^{i\gamma/2}) = \frac{1}{|g'(e^{i\gamma/2})|} = \frac{2\gamma}{\pi}.$$

Therefore,

$$\frac{2d_{S_\gamma}(e^{i\gamma/2})}{r_{S_\gamma}(e^{i\gamma/2})} = \frac{\pi \sin(\gamma/2)}{\gamma}.$$

If γ tends to $\pi-$, then S_γ tends to the half-plane and $2d_{S_\gamma}(e^{i\gamma/2})/r_{S_\gamma}(e^{i\gamma/2}) \rightarrow 1$. If we allow $\gamma \rightarrow 0+$, then $2d_{S_\gamma}(e^{i\gamma/2})/r_{S_\gamma}(e^{i\gamma/2})$ tends to $\pi/2$; this constant corresponds to the case of the strip $\{z : |\operatorname{Im} z| < \pi/2\}$.

Example 6.5. Isosceles triangles: for an arbitrary triangle Δ , the upper bound of $2d_\Delta(\cdot)/r_\Delta(\cdot)$ is attained at the center of its inscribed circle. Here, we describe the situation for the case of an isosceles triangle Δ_α with vertices $1, \pm i \cot(\alpha\pi)$ and angles $(1 - 2\alpha)\pi, \alpha\pi$, and $\alpha\pi$, where $\alpha \in (0, 1/2)$. The conformal mapping of the right half-plane onto Δ_α is given by

$$G_\alpha(z) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(1-\alpha)}{\Gamma(1/2-\alpha)} \int_0^z \frac{dt}{(1+t^2)^{1-\alpha}} = \frac{2z}{\sqrt{\pi}} \frac{\Gamma(1-\alpha)}{\Gamma(1/2-\alpha)} {}_2F_1\left(\frac{1}{2}, 1-\alpha; \frac{3}{2}; -z^2\right).$$

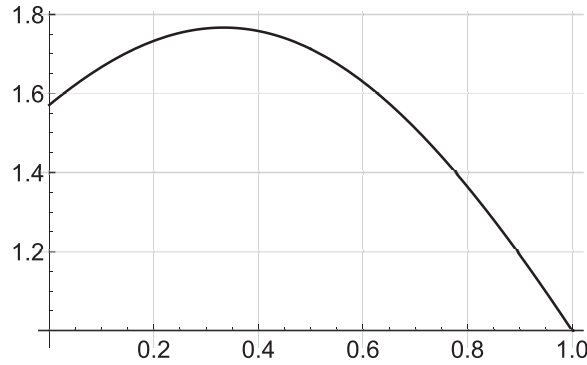


FIGURE 6 The graph of $M_3(\alpha)$, where $\alpha \in (0, 1)$.

TABLE 1 Some values of $M_1(\alpha)$, where $\alpha \in (0, 1/2)$.

α	$M_1(\alpha)$
$\rightarrow 0+$	$\rightarrow (\pi/2) = 1.5707963267 \dots$
$1/12$	$1.6527823736 \dots$
$1/6$	$1.7147087445 \dots$
$1/4$	$1.7534053628 \dots$
$5/12$	$1.7531525258 \dots$
$\rightarrow (1/2)-$	$\rightarrow (\pi/2)\text{cth}(\pi/2) = 1.7126885749 \dots$

Here, ${}_2F_1$ is the Gaussian hypergeometric function [3, Ch 1]. The center of the inscribed circle for Δ_α is at the point $(1 - \tan^2(\alpha\pi/2))/2$ and its preimage $x = x(\alpha)$ under the mapping G_α can be found from the equation

$$\frac{2x}{\sqrt{\pi}} \frac{\Gamma(1-\alpha)}{\Gamma(1/2-\alpha)} {}_2F_1\left(\frac{1}{2}, 1-\alpha; \frac{3}{2}; -x^2\right) = \frac{1}{2} \left(1 - \tan^2 \frac{\alpha\pi}{2}\right).$$

Then, the maximal value of $2d_{\Delta_\alpha}(\cdot)/r_{\Delta_\alpha}(\cdot)$ is equal to

$$M_1(\alpha) := \frac{\sqrt{\pi}(1 - \tan^2(\alpha\pi/2))\Gamma(1/2-\alpha)(1+x(\alpha)^2)^{1-\alpha}}{4\Gamma(1-\alpha)x(\alpha)}.$$

From the graph of $M_1(\alpha)$ (see Figure 6 and Remark 6.8), we can see that the maximal value of $M_1(\alpha)$ is attained at the point $\alpha_0 = 1/3$ corresponding to the case of regular triangle; possibly this can be proved strictly analytically. The corresponding value is

$$M_1(\alpha_0) = \frac{2^{1/3}\sqrt{\pi}\Gamma(1/6)}{3\sqrt{3}\Gamma(2/3)} = 1.76663875 \dots$$

In Table 1, we give some values of $M_1(\alpha)$, where $\alpha \in (0, 1/2)$.

We note that $\alpha \rightarrow 0+$ gives the case of a strip and $\alpha \rightarrow (1/2)-$ gives the case of a half-strip, see the next example.

Example 6.6. Half-strip $\Omega_a := \{z : |\operatorname{Re} z| < a \text{ and } \operatorname{Im} z > 0, 0 < a \in \mathbb{R}\}$. The upper estimate $2d_{\Omega_a}(\cdot)/r_{\Omega_a}(\cdot)$ is attained at the intersection point of the bisectors of the two angles at its two finite vertices. The function mapping the half-strip Ω_a onto the upper half-plane is $g(z) = \sin(\pi z/2a)$. The intersection point of the bisectors is ia . It follows from Equation (5) that

$$r_{\Omega_a}(ia) = \frac{2 \operatorname{Im} g(ia)}{|g'(ia)|} = \frac{2\operatorname{sh}(\pi/2)}{(\pi/2a)\operatorname{ch}(\pi/2)} = \frac{4a}{\pi} \operatorname{th}\left(\frac{\pi}{2}\right).$$

TABLE 2 Some values of $M_2(\alpha)$, where $\alpha \in (1/2, 1)$.

α	$M_2(\alpha)$
2/3	1.554662095759 ...
3/4	1.441224578770 ...
5/6	1.308765658869 ...

Therefore,

$$\frac{2d_{\Omega_\alpha}(ia)}{r_{\Omega_\alpha}(ia)} = \frac{\pi}{2} \operatorname{cth}\left(\frac{\pi}{2}\right) =: M_0 \approx 1.712689.$$

Example 6.7. Arbitrary convex triangle with a vertex at infinity. Let Λ_α be a symmetric triangle bounded with the segment $[-i, i]$ and two rays going from the points $\pm i$ and forming the angles $\alpha\pi$ with the segment, where $\alpha \in (1/2, 1)$. The upper estimate for $2d_{\Lambda_\alpha}(\cdot)/r_{\Lambda_\alpha}(\cdot)$ is attained at the intersection point of the bisectors of the two angles at its two finite vertices. The function P_α mapping the right half-plane onto Λ_α has the form

$$P_\alpha(z) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(1/2 + \alpha)}{\Gamma(\alpha)} \int_0^z \frac{dt}{(1+t^2)^{1-\alpha}} = \frac{2z}{\sqrt{\pi}} \frac{\Gamma(1/2 + \alpha)}{\Gamma(\alpha)} {}_2F_1\left(\frac{1}{2}, 1 - \alpha; \frac{3}{2}; -z^2\right).$$

The intersection point of the bisectors of the angles at the finite vertices is at the point $\tan(\alpha\pi/2)$, therefore, we can find $x = x(\alpha)$ which is the preimage of $\tan(\alpha\pi/2)$ under the mapping P_α from the equation

$$\frac{2x}{\sqrt{\pi}} \frac{\Gamma(1/2 + \alpha)}{\Gamma(\alpha)} {}_2F_1\left(\frac{1}{2}, 1 - \alpha; \frac{3}{2}; -x^2\right) = \tan\left(\frac{\alpha\pi}{2}\right).$$

Then, the maximal value of $2d_{\Lambda_\alpha}(\cdot)/r_{\Lambda_\alpha}(\cdot)$ is equal to

$$M_2(\alpha) := \frac{\sqrt{\pi} \tan(\alpha\pi/2) \Gamma(\alpha) (1 + x(\alpha)^2)^{1-\alpha}}{2\Gamma(1/2 + \alpha)x(\alpha)}.$$

Some values of $M_2(\alpha)$ are also given in Table 2:

Remark 6.8. We can combine Examples 6.5 and 6.7 above by defining $M_3(\alpha)$ as follows for all $\alpha \in (0, 1)$:

$$M_3(\alpha) := \begin{cases} M_1(\alpha), & 0 < \alpha < \frac{1}{2}; \\ M_0, & \alpha = \frac{1}{2}; \\ M_2(\alpha), & \frac{1}{2} < \alpha < 1. \end{cases}$$

Figure 6 shows the graph of $M_3(\alpha)$ for $\alpha \in (0, 1)$.

Example 6.9. Rhomb: let Π be a rhomb with acute angle δ and length of side 1. The upper estimate of $2d_\Pi(\cdot)/r_\Pi(\cdot)$ is attained at its center. From [15, Chap. 1, Section 1.22] we have that the conformal radius at its center is equal to

$$r_\Pi(0) = \frac{4\sqrt{\pi}}{\Gamma\left(\frac{\delta}{2\pi}\right)\Gamma\left(\frac{\pi-\delta}{2\pi}\right)}.$$

Since the distance from the center to the boundary $d_\Pi(0)$ is equal $(1/2) \sin \delta$, we obtain

$$\frac{2d_\Pi(0)}{r_\Pi(0)} = \frac{\sin \delta}{4\sqrt{\pi}} \Gamma\left(\frac{\delta}{2\pi}\right)\Gamma\left(\frac{\pi-\delta}{2\pi}\right) = \frac{\pi\sqrt{\pi}}{2\Gamma\left(\frac{2\pi-\delta}{2\pi}\right)\Gamma\left(\frac{\pi+\delta}{2\pi}\right)} =: \phi(\delta).$$

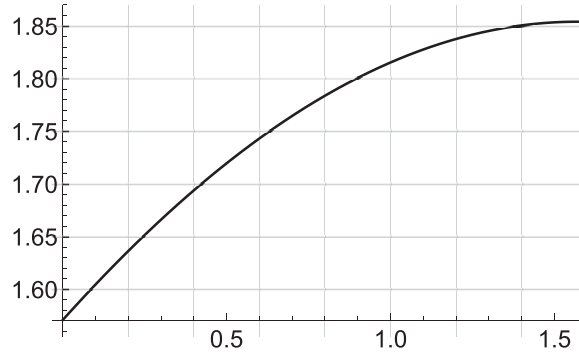


FIGURE 7 The graph of $\phi(\delta)$.

This value tends to

$$\frac{\Gamma^2(1/4)}{4\sqrt{\pi}} = 1.854074677 \dots,$$

(the case of a square, cf. Equation (15)), as $\delta \rightarrow \pi/2$, and to $\pi/2$ (the case of a strip), as $\delta \rightarrow 0$. It is easy to see that the function $\phi(\delta)$ is positive on $(0, \pi/2)$ and

$$\frac{\phi'(\delta)}{\phi(\delta)} = \frac{1}{2\pi} \left[\psi\left(\frac{2\pi - \delta}{2\pi}\right) - \psi\left(\frac{\pi + \delta}{2\pi}\right) \right],$$

where $\psi(x) = \frac{d}{dx} \log \Gamma(x)$ is the digamma function. Since $\psi(x)$ increases on $(0, +\infty)$ and $(2\pi - \delta)/(2\pi) > (\pi + \delta)/(2\pi)$, $0 < \delta < \pi/2$ we see that $\phi(\delta)$ increases on $(0, \pi/2)$.

The graph of $\phi(\delta)$ is given in Figure 7.

Example 6.10. Trapezoid: let us denote an arbitrary trapezoid by T .

- For long trapezoids, the maximum of $2d_T(\cdot)/r_T(\cdot)$ is attained at its midsegment, more precisely, between the intersection points of bisectors of the angles at two adjacent vertices, lying on different parallel sides. Most likely, the maximum is attained at one of the intersection points of its bisectors. This case requires additional investigation.
- For short trapezoids, the maximum of $2d_T(\cdot)/r_T(\cdot)$ is attained at the segment joining the intersection points of bisectors of the angles at two adjacent vertices, lying on different parallel sides. This segment is on the bisector of the angle formed by the straight lines, containing the nonparallel sides, of the trapezoid. Most likely, the maximum is attained at one of the intersection points of the bisectors described above. This case also requires additional investigation.
- The transitional case is when the considered trapezoid has an inscribed circle. Then, the upper estimate is attained at its center.

Example 6.11. Arbitrary convex quadrilateral: the situations are close to the case of trapezoids. The only difference that, in the non-trapezoidal case, instead of a part of the midsegment we consider the segment of the bisector of the angle, which forms the straight lines, containing the corresponding pair of nonparallel sides.

Example 6.12. Regular n -gon: let N be a regular n -gon with the side length a . The upper estimate of $2d_N(\cdot)/r_N(\cdot)$ is attained at its center. Again, from [15, Chap. 1, Section 1.22] we have that the conformal radius at its center is equal to

$$r_N(0) = \frac{na\Gamma\left(1 - \frac{1}{n}\right)}{2^{1-\frac{2}{n}}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2} - \frac{1}{n}\right)}.$$

The distance from the center to the boundary is equal to the radius of the inscribed circle:

$$d_N(0) = \frac{a}{2 \tan \frac{\pi}{n}}.$$

Thus, we obtain

$$\frac{2d_N(0)}{r_N(0)} = \frac{2^{1-\frac{2}{n}} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} - \frac{1}{n}\right)}{n \tan \frac{\pi}{n} \Gamma\left(1 - \frac{1}{n}\right)} = \frac{1}{n} 2^{1-\frac{2}{n}} B\left(\frac{1}{n}, \frac{1}{2}\right),$$

where $B(\cdot, \cdot)$ is the well-known Euler beta function. This value grows as n increases and tends to 2, as n tends to infinity (the case of a circle). For n equals to 3, 4, and 6 this formula gives values for equilateral triangle, square, and regular hexagon, respectively.

Example 6.13. Polygons having inscribed circles: the upper estimate is attained at the center of the inscribed circle.

Now, we will estimate the ratio $2d_{D_1}(\cdot)/r_{D_1}(\cdot)$ and $2d_{D_2}(\cdot)/r_{D_2}(\cdot)$ for non-polygonal domains described in Examples 5.1 and 5.2, respectively.

Example 6.14. The domain D_1 which is the image of the unit disk under the mapping $w = \phi_1(z) = \sqrt{1+z} - 1$ is convex. From Lemma 4.2, it follows that the maximum of the ratio $2d_{D_1}(w)/r_{D_1}(w)$, $w \in D_1$, is attained at some point w which is real, that is, $-1 < w < \sqrt{2} - 1$. Now, we will find $d_{D_1}(w)$ for such w .

Denote $t = w + 1$, $0 < t < \sqrt{2}$. Then

$$\sqrt{1 + e^{i\theta}} = e^{i\theta/4} \sqrt{2 \cos(\theta/2)}, \quad -\pi/2 \leq \theta \leq \pi/2,$$

and

$$A(\theta) := |\phi_1(e^{i\theta}) - w|^2 = |\sqrt{1 + e^{i\theta}} - t|^2 = t^2 - 2\sqrt{2}t \cos(\theta/4) \sqrt{\cos(\theta/2)} + 2 \cos(\theta/2).$$

If we put $\tau = \cos(\theta/4)$, then we obtain $A(\theta) = f(\tau)$, where

$$f(\tau) = t^2 - 2\sqrt{2}t\tau \sqrt{2\tau^2 - 1} + 2(2\tau^2 - 1), \quad \sqrt{2}/2 \leq |\tau| \leq 1.$$

Analyzing $f(\tau)$, we find that for $2\sqrt{2}/3 \leq t < \sqrt{2}$ we have $f(\tau) \geq f(1) = (\sqrt{2} - t)^2$, therefore, the distance from w to the boundary of D_1 , that is, $d_{D_1}(w)$, equals to $\sqrt{2} - t$. If $0 < t < 2\sqrt{2}/3$, then $f(\tau)$ attains its minimum at the point τ_0 such that

$$\tau_0^2 = \frac{1}{4} \left(1 + \frac{1}{\sqrt{1-t^2}} \right), \quad (16)$$

and consequently $d_{D_1}(w) = \sqrt{\sqrt{1-t^2} - (1-t^2)}$. Therefore,

$$d_{D_1}(w) = \begin{cases} \sqrt{\sqrt{1-t^2} - (1-t^2)}, & 0 < t < 2\sqrt{2}/3; \\ \sqrt{2} - t, & 2\sqrt{2}/3 \leq t < \sqrt{2}, \end{cases} \quad (17)$$

where $t = 1 + w$.

Let $\tau_0 = \cos(\theta_0/4)$. Analyzing Equation (16), we see that when t increases from 0 to $2\sqrt{2}/3$ the point θ_0 decreases from π to 0. Therefore, for every boundary point $\phi_1(\theta)$, $0 < \theta < \pi$, there is $t \in [0, 2\sqrt{2}/3]$ such that the maximal circle in D_1 centered at the point $w = t - 1$ touch $\phi_1(\theta)$. Because of the symmetry of D_1 with respect to the real axis, a similar fact is valid for $-\pi < \theta < 0$. Then from Lemma 4.2 we conclude that the maximal value of $2d_{D_1}(w)/r_{D_1}(w)$, $w \in D_1$, is attained at some point of the real axis.

For the conformal radius at the points $w = \phi_1(z)$, $-1 < z < 1$, we have $r_{D_1}(w) = (1/2)(1-z)^{1/2}(1+z)$. Taking into account that $z = (w+1)^2 - 1 = t^2 - 1$ we have

$$r_{D_1}(w) = (1/2)t(2-t^2). \quad (18)$$

Simple analysis of the function $2d_{D_1}(w)/r_{D_1}(w)$ with the help of Equations (17) and (18) shows that it has a unique maximum at the point $w_0 = \sqrt{1-u_0^2} - 1 = -0.109718 \dots$, where $u_0 = 0.45541 \dots$ is a unique root of the cubic equation $4u^3 + 3u^2 - 1 = 0$. The maximum equals $1.85318 \dots$. Therefore, in D_1 we have the sharp inequality

$$\frac{2d_{D_1}(w)}{r_{D_1}(w)} \leq 1.85318 \dots, \quad w \in D_1.$$

Example 6.15. The domain, D_2 which is the image of the unit disk under the mapping $w = \phi_2(z) = z + \sqrt{1+z^2} - 1$. We will also find the maximum of the ratio $2d_{D_2}(w)/r_{D_2}(w)$ but, because of the fact that this domain is not convex, we will confine ourselves to considering the case of real w .

For real $w = \phi_2(z)$, we have

$$r_{D_2}(w) = \frac{z + \sqrt{1+z^2}}{\sqrt{1+z^2}}(1-z^2).$$

Since the disk $\{|w - (\sqrt{2}-1)| < 1\}$ is a subset of D_2 , and it contains two of its boundary points $\sqrt{2}-2$ and $\sqrt{2}$, we have for $\sqrt{2}-2 < w < \sqrt{2}$:

$$d_{D_2}(w) = \begin{cases} w - (\sqrt{2}-2), & \sqrt{2}-2 < w < \sqrt{2}-1; \\ \sqrt{2}-w, & \sqrt{2}-1 \leq w < \sqrt{2}. \end{cases}$$

If we denote $t = w + 1$, then

$$r_{D_2}(w) = \frac{6t^2 - t^4 - 1}{2(t^2 + 1)},$$

$$d_{D_2}(w) = \begin{cases} t - (\sqrt{2}-1), & \sqrt{2}-1 < t < \sqrt{2}; \\ \sqrt{2} + 1 - t, & \sqrt{2} \leq t < \sqrt{2} + 1 \end{cases}$$

and

$$\frac{2d_{D_2}(w)}{r_{D_2}(w)} = B(t) := \begin{cases} \frac{(t - (\sqrt{2}-1))(t^2 + 1)}{6t^2 - t^4 - 1}, & \sqrt{2}-1 < t < \sqrt{2}; \\ \frac{((\sqrt{2}+1) - t)(t^2 + 1)}{6t^2 - t^4 - 1}, & \sqrt{2} \leq t < \sqrt{2} + 1. \end{cases} \quad (19)$$

From Equation (19), it is easy to see that $B(t)$ increases on $[\sqrt{2}-1, \sqrt{2}]$ and decreases on $[\sqrt{2}, \sqrt{2}+1]$. Therefore, the maximum is attained at $t = \sqrt{2}$ and is equal to $B(\sqrt{2}) = 12/7 = 1.71429 \dots$

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

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